

Stability of the Mann and Ishikawa Iteration Procedures for ϕ -Strong Pseudocontractions and Nonlinear Equations of the ϕ -Strongly Accretive Type

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We study the stability of the Mann and Ishikawa iteration procedures for the class of Lipschitz ϕ -strongly pseudocontractive maps in arbitrary real Banach spaces. As a consequence we study the stability of these iteration procedures for the iterative approximation of solutions of nonlinear equations of the ϕ -strongly accretive type. Furthermore, we prove that the T -stability of the Mann iteration procedure leads to the strong convergence (to the fixed point of T) of the so-called Mann iteration method with errors introduced by Liu [*J. Math. Anal. Appl.* **194** (1995), 114–125]. © 1998 Academic Press

1. INTRODUCTION

Suppose E is a real Banach space and T is a self-map of E . Suppose $x_0 \in E$ and $x_{n+1} = f(T, x_n)$ defines an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^\infty$ in E . For example, the function iteration $x_{n+1} = f(T, x_n) = Tx_n$. Suppose $F(T) = \{x \in E: Tx = x\} \neq \emptyset$ and that $\{x_n\}$ converges strongly to $x^* \in F(T)$. Suppose $\{y_n\}_{n=0}^\infty$ is a sequence in E and $\{\epsilon_n\}_{n=0}^\infty$ is a sequence in $\mathfrak{R}^+ = [0, \infty)$ given by $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$. If $\lim_{n \rightarrow \infty} \epsilon_n = 0$ implies that $\lim_{n \rightarrow \infty} y_n = x^*$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T -stable or stable with respect to T (see, for example, [10–12, 17, 18, 20–22, 24–26]).

We say that the iteration procedure $\{x_n\}$ is almost T stable or almost stable with respect to T if $\sum_{n=0}^\infty \epsilon_n < \infty$ implies that $\lim y_n = x^*$. Clearly an

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iteration procedure $\{x_n\}$ which is T -stable is almost T -stable. We shall show later in this paper that an iteration procedure which is almost T -stable may fail to be T -stable. We shall also present an example where some iteration procedures are neither T -stable nor almost T -stable.

Stability results for several iteration procedures for certain classes of nonlinear mappings have been established in recent papers by several authors (see, for example, [10–12, 17, 18, 20–22, 24–26]. Harder and Hicks [12] showed how such sequences $\{y_n\}$ could arise in practice and demonstrated the importance of investigating the stability of various iteration procedures for various classes of nonlinear mappings. As was remarked by Massa (*Math. Reviews* **90a** (1990), no. 54109a, 54H25), the discussion about stability is very rich in examples. In [10] some applications of stability results to first order differential equations are discussed.

Recently, the author [19, 22] studied the stability of certain *Mann* [16] and *Ishikawa* [13] iteration procedures for fixed points of *Lipschitz strong pseudocontractions*, and solutions of nonlinear *accretive operator* equations.

Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single-valued. In the sequel we shall denote the single-valued normalized duality mapping by j .

An operator T with domain $D(T)$ and range $R(T)$ in E is called a *strong pseudocontraction* if for all $x, y \in D(T)$ there exist $j(x - y) \in J(x - y)$ and $t > 1$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \frac{1}{t} \|x - y\|^2. \quad (1)$$

T is called *ϕ -strongly pseudocontractive* (see, for example, [19]) if for all $x, y \in D(T)$ there exist $j(x - y) \in J(x - y)$ and a strictly increasing function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|) \|x - y\|. \quad (2)$$

It is shown in [19] that the class of strongly pseudocontractive operators is a proper subset of the class of ϕ -strongly pseudocontractive operators. The class of strong pseudocontractions and the class of ϕ -strong pseudocontractions have been studied extensively by several authors (see, for example, [1–9, 15, 19, 20, 22, 23, 28]). Interest in strong pseudocontractions and ϕ -strong pseudocontractions stems mainly from their firm connection with the important classes of *strongly accretive operators* and *ϕ -strongly accretive operators*, respectively.

An operator T is called *strongly accretive* if for all $x, y \in D(T)$ there exist $j(x - y) \in J(x - y)$ and a constant $k > 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq k\|x - y\|^2. \quad (3)$$

T is called *ϕ -strongly accretive* if for all $x, y \in D(T)$ there exist $j(x - y) \in J(x - y)$ and a strictly increasing function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x, y) \rangle \geq \phi(\|x - y\|)\|x - y\|. \quad (4)$$

If I denotes the identity operator, then it follows from inequalities (1)–(4) that T is strongly pseudocontractive (respectively, ϕ -strongly pseudocontractive) if and only if $(I - T)$ is strongly accretive (respectively, ϕ -strongly accretive). Thus the mapping theory for strongly accretive operators (respectively, ϕ -strongly accretive operators) is closely related to the fixed point theory for strongly pseudocontractive operators (respectively, ϕ -strongly pseudocontractive operators). Recent interest in mapping theory for strongly accretive operators and ϕ -strongly accretive operators, particularly as it relates to the existence theorems for nonlinear ordinary and partial differential equations, has prompted a corresponding interest in fixed-point theory for strong pseudocontractions and ϕ -strong pseudocontractions (see, for example, [1–9, 14, 15, 19, 20, 22, 23, 28]).

It is well known (see, for example, [15]) that if $T: E \rightarrow E$ is continuous and strongly pseudocontractive, then T has a unique fixed point. Furthermore, if $T: E \rightarrow E$ is continuous and strongly accretive, then T is surjective, so that for a given $f \in E$, the equation

$$Tx = f \quad (5)$$

has a unique solution.

In [21] the author proved that certain Mann and Ishikawa iteration procedures are stable with respect to Lipschitz strong pseudocontractions in real q -uniformly smooth Banach spaces. As a consequence of our results we proved that certain Mann and Ishikawa iteration procedures for approximating the solution of (5) (when it exists) are stable in real q -uniformly smooth Banach spaces. In [22] we extended the results of [21] to arbitrary real Banach spaces.

It is our purpose in this paper to examine the stability of the Mann and Ishikawa iteration procedures for the more general class of Lipschitz ϕ -strong pseudocontractions in arbitrary real Banach spaces. As a consequence, we examine, in arbitrary real Banach spaces, the stability of certain Mann and Ishikawa iteration methods for the iterative approximation of the solution of (5) (when it exists) when T is a Lipschitz ϕ -strongly

accretive operator. Furthermore, we prove that the T -stability or the almost T -stability of the Mann iteration procedure leads to the strong convergence of the so-called *Mann iteration method with errors* introduced in [15] to the fixed point of T .

We shall need the following:

LEMMA TX [27, p. 303]. *Suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of nonnegative numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.

2. MAIN RESULTS

For the rest of this paper L will denote the Lipschitz constant of T and $L_* = 1 + L$. We now prove the following:

THEOREM 1. *Suppose E is an arbitrary real Banach space and $T: E \rightarrow E$ is a Lipschitz ϕ -strongly pseudocontractive operator. Suppose $F(T) \neq \emptyset$ and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences satisfying the conditions*

- (i) $0 \leq \alpha_n, \beta_n \leq 1, n \geq 0$,
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- (iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$,
- (iv) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Suppose $\{x_n\}_{n=0}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in E$ by

$$\begin{aligned} z_n &= (1 - \beta_n)x_n + \beta_n T x_n, & n \geq 0, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T z_n, & n \geq 0. \end{aligned}$$

Suppose $\{y_n\}_{n=0}^{\infty}$ is a sequence in E and define $\{\epsilon_n\}_{n=0}^{\infty} \subseteq \mathfrak{R}^+$ by

$$\begin{aligned} s_n &= (1 - \beta_n)y_n + \beta_n T y_n, & n \geq 0, \\ \epsilon_n &= \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T s_n\|, & n \geq 0. \end{aligned}$$

Then:

1. *The sequence $\{x_n\}$ converges strongly to the fixed point x^* of T .*

$$\begin{aligned} 2. \quad \|y_{n+1} - x^*\| &\leq [1 - \alpha_n r(p_n, x^*)] \|y_n - x^*\| \\ &\quad + [L^3 + 4L^2 + 3(1 + L)] \alpha_n^2 \|y_n - x^*\| \\ &\quad + L(1 + L) \alpha_n \beta_n \|y_n - x^*\| + \epsilon_n, \end{aligned} \tag{6}$$

where $p_n = (1 - \alpha_n)y_n + \alpha_n Ts_n$, and

$$r(p_n, x^*) = \frac{\phi(\|p_n - x^*\|)}{1 + \phi(\|p_n - x^*\|) + \|p_n - x^*\|}.$$

3. $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies $\lim_{n \rightarrow \infty} y_n = x^*$, so that $\{x_n\}$ is almost T -stable.

4. $\lim_{n \rightarrow \infty} y_n = x^*$ implies $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

Proof. It follows from inequality (2) that if T has a fixed point, then the fixed point is unique. Let x^* denote the fixed point.

Item 1 is a consequence of Theorem 2 of [23].

We now prove 2–4. Observe that inequality (2) implies that

$$\begin{aligned} & \langle (I - T)x - (I - T)y, j(x - y) \rangle \\ & \geq \phi(\|x - y\|) \|x - y\| \\ & \geq \frac{\phi(\|x - y\|)}{1 + \phi(\|x - y\|) + \|x - y\|} \|x - y\|^2 = r(x, y) \|x - y\|^2, \quad (7) \end{aligned}$$

where

$$r(x, y) = \frac{\phi(\|x - y\|)}{1 + \phi(\|x - y\|) + \|x - y\|} \in [0, 1) \quad \forall x, y \in E.$$

It follows from (7) that

$$\langle (I - T)x - r(x, y)x - ((I - T)y - r(x, y)y), j(x - y) \rangle \geq 0,$$

so that it follows from Lemma 1.1 of Kato [14] that

$$\|x - y\| \leq \|x - y + \lambda[(I - T)x - r(x, y)x - ((I - T)y - r(x, y)y)]\| \quad (8)$$

for all $x, y \in E$ and for all $\lambda > 0$. Observe that

$$\begin{aligned} \|y_{n+1} - x^*\| & \leq \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Ts_n\| \\ & \quad + \|(1 - \alpha_n)(y_n - x^*) + \alpha_n(Ts_n - x^*)\| \\ & = \epsilon_n + \|(1 - \alpha_n)(y_n - x^*) + \alpha_n(Ts_n - x^*)\|. \quad (9) \end{aligned}$$

Set $p_n = (1 - \alpha_n)y_n + \alpha_n Ts_n$. Then

$$\begin{aligned} y_n & = p_n + \alpha_n y_n - \alpha_n Ts_n \\ & = (1 + \alpha_n)p_n + \alpha_n [(I - T)p_n - r(p_n, x^*)p_n] \\ & \quad - (1 - r(p_n, x^*))\alpha_n y_n \\ & \quad + (2 - r(p_n, x^*))\alpha_n^2 (y_n - Ts_n) + \alpha_n (Tp_n - Ts_n). \end{aligned}$$

Observe that

$$x^* = (1 + \alpha_n)x^* + \alpha_n[(I - T)x^* - r(p_n, x^*)x^*] \\ - (1 - r(p_n, x^*))\alpha_n x^*,$$

so that

$$y_n - x^* = (1 + \alpha_n) \left[p_n - x^* + \frac{\alpha_n}{1 + \alpha_n} [(I - T)p_n - r(p_n, x^*)p_n \right. \\ \left. - ((I - T)x^* - r(p_n, x^*)x^*)] \right] \\ - (1 - r(p_n, x^*))\alpha_n(y_n - x^*) + (2 - r(p_n, x^*)) \\ \times \alpha_n^2(y_n - Ts_n) + \alpha_n(Tp_n - Ts_n).$$

Thus

$$\|y_n - x^*\| \geq (1 + \alpha_n) \left\| p_n - x^* + \frac{\alpha_n}{1 + \alpha_n} [(I - T)p_n - r(p_n, x^*)p_n \right. \\ \left. - ((I - T)x^* - r(p_n, x^*)x^*)] \right\| \\ - (1 - r(p_n, x^*))\alpha_n\|y_n - x^*\| - (2 - r(p_n, x^*)) \\ \times \alpha_n^2\|y_n - Ts_n\| - \alpha_n\|Tp_n - Ts_n\| \\ \geq (1 + \alpha_n)\|p_n - x^*\| - (1 - r(p_n, x^*))\alpha_n\|y_n - x^*\| \\ - (2 - r(p_n, x^*))\alpha_n^2\|y_n - Ts_n\| - \alpha_n\|Tp_n - Ts_n\| \\ \text{(using (8)).}$$

Hence

$$\|p_n - x^*\| \leq \frac{[1 + (1 - r(p_n, x^*))\alpha_n]}{1 + \alpha_n} \|y_n - x^*\| \\ + (2 - r(p_n, x^*))\alpha_n^2\|y_n - Ts_n\| + \alpha_n\|Tp_n - Ts_n\|. \quad (10)$$

Furthermore, we have the estimates

$$\begin{aligned} \|s_n - x^*\| &= \|(1 - \beta_n)(y_n - x^*) + \beta_n(Ty_n - x^*)\| \\ &\leq [1 - \beta_n + L\beta_n]\|y_n - x^*\| \leq (1 + L)\|y_n - x^*\|, \end{aligned} \quad (11)$$

$$\begin{aligned} \|y_n - Ts_n\| &\leq \|y_n - x^*\| + L\|s_n - x^*\| \leq [1 + L(1 + L)]\|y_n - x^*\|, \\ \|Tp_n - Ts_n\| &\leq L\|p_n - s_n\| \\ &= L\|(1 - \alpha_n)(y_n - s_n) + \alpha_n(Ts_n - s_n)\| \\ &\leq L(1 - \alpha_n)\beta_n\|y_n - Ty_n\| + \alpha_nL(1 + L)\|s_n - x^*\| \\ &\leq [L(1 + L)\beta_n + \alpha_nL(1 + L)^2]\|y_n - x^*\|. \end{aligned} \quad (12)$$

Using (11) and (12) in (10) we obtain

$$\begin{aligned} \|p_n - x^*\| &\leq \frac{[1 + (1 - r(p_n, x^*))\alpha_n]}{1 + \alpha_n}\|y_n - x^*\| + (2 - r(p_n, x^*)) \\ &\quad \times \alpha_n^2(1 + L(1 + L))\|y_n - x^*\| \\ &\quad + \alpha_n[L(1 + L)\beta_n + \alpha_nL(1 + L)^2]\|y_n - x^*\| \\ &\leq \frac{[1 + (1 - r(p_n, x^*))\alpha_n]}{1 + \alpha_n}\|y_n - x^*\| \\ &\quad + \alpha_n^2[2(1 + L(1 + L)) + L(1 + L)^2]\|y_n - x^*\| \\ &\quad + \alpha_n\beta_nL(1 + L)\|y_n - x^*\| \\ &\leq [1 + (1 - r(p_n, x^*))\alpha_n][1 - \alpha_n + \alpha_n^2]\|y_n - x^*\| \\ &\quad + [L^3 + 4L^2 + 3L + 2]\alpha_n^2\|y_n - x^*\| \\ &\quad + \alpha_n\beta_nL(1 + L)\|y_n - x^*\| \\ &\leq [1 - \alpha_nr(p_n, x^*) + \alpha_n^2]\|y_n - x^*\| \\ &\quad + [L^3 + 4L^2 + 3L + 2]\alpha_n^2\|y_n - x^*\| \\ &\quad + \alpha_n\beta_nL(1 + L)\|y_n - x^*\| \\ &= [1 - \alpha_nr(p_n, x^*)]\|y_n - x^*\| \\ &\quad + [L^3 + 4L^2 + 3(1 + L)]\alpha_n^2\|y_n - x^*\| \\ &\quad + \alpha_n\beta_nL(1 + L)\|y_n - x^*\|. \end{aligned} \quad (13)$$

Using (13) in (9) now yields (6), completing the proof of 2.

Next we prove 3. Suppose $\sum_{n=0}^{\infty} \epsilon_n < \infty$. Then (6) implies that

$$\|y_{n+1} - x^*\| \leq [1 + \delta_n] \|y_n - x^*\| + \epsilon_n, \quad (14)$$

where $\delta_n = (L^3 + 4L^2 + 3L + 3)\alpha_n^2 + \alpha_n \beta_n L(1 + L)$. Since $\sum_{n=0}^{\infty} \delta_n < \infty$, inequality (14) implies that $\{\|y_n - x^*\|\}_{n=0}^{\infty}$ is bounded. Suppose $\|y_n - x^*\| \leq D \forall n \geq 0$. Then (14) implies that

$$\|y_{n+1} - x^*\| \leq \|y_n - x^*\| + D\delta_n + \epsilon_n = \|y_n - x^*\| + \sigma_n,$$

where $\sigma_n = D\delta_n + \epsilon_n$, and it follows from Lemma TX that $\lim_{n \rightarrow \infty} \|y_n - x^*\|$ exists.

Suppose $\lim_{n \rightarrow \infty} \|y_n - x^*\| = \delta \geq 0$. We prove that $\delta = 0$. Assume, for contradiction, that $\delta > 0$. Since $\|Ts_n - x^*\| \leq L(1 + L)\|y_n - x^*\| \leq L(1 + L)D$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, it follows that $\lim_{n \rightarrow \infty} \alpha_n \|Ts_n - x^*\| = 0$. Furthermore, it follows from the inequality

$$\begin{aligned} (1 - \alpha_n)\|y_n - x^*\| - \alpha_n \|Ts_n - x^*\| \\ \leq \|p_n - x^*\| \leq (1 - \alpha_n)\|y_n - x^*\| + \alpha_n \|Ts_n - x^*\| \end{aligned}$$

that $\lim_{n \rightarrow \infty} \|p_n - x^*\| = \delta > 0$. Hence there exists a nonnegative integer N such that

$$\|p_n - x^*\| \geq \frac{\delta}{2}, \quad \|y_n - x^*\| \geq \frac{\delta}{2} \quad \forall n \geq N.$$

Since

$$\begin{aligned} \|p_n - x^*\| &\leq (1 - \alpha_n)\|y_n - x^*\| + \alpha_n L(1 + L)\|y_n - x^*\| \\ &\leq [1 + L(1 + L)]D := M, \end{aligned}$$

then

$$r(p_n, x^*) = \frac{\phi(\|p_n - x^*\|)}{1 + \phi(\|p_n - x^*\|) + \|p_n - x^*\|} \geq \frac{\phi(\delta/2)}{1 + \phi(M) + M} \quad \forall n \geq N,$$

so that (6) implies that

$$\begin{aligned} \|y_{n+1} - x^*\| &\leq \left[1 - \frac{\phi(\delta/2)\alpha_n}{1 + \phi(M) + M} \right] \|y_n - x^*\| \\ &\quad + [L^3 + 4L^2 + 3L + 3]\alpha_n^2 \|y_n - x^*\| \\ &\quad + \alpha_n \beta_n L(1 + L)\|y_n - x^*\| + \epsilon_n \quad (\forall n \geq N) \\ &\leq \|y_n - x^*\| - \frac{\phi(\delta/2)(\delta/2)\alpha_n}{1 + \phi(M) + M} \\ &\quad + [L^3 + 4L^2 + 3L + 3]D\alpha_n^2 \\ &\quad + \alpha_n \beta_n L(1 + L)D + \epsilon_n \quad (\forall n \geq N). \end{aligned}$$

Thus

$$\frac{\phi(\delta/2)(\delta/2)\alpha_n}{1 + \phi(M) + M} \leq \|y_n - x^*\| - \|y_{n+1} - x^*\| + \rho_n \quad \forall n \geq N, \quad (15)$$

where $\rho_n = [L^3 + 4L^2 + 3L + 3]D\alpha_n^2 + \alpha_n \beta_n L(1 + L)D + \epsilon_n$. It follows from (15) that

$$\frac{\phi(\delta/2)\delta/2}{1 + \phi(M) + M} \sum_{j=N}^n \alpha_j \leq \|y_N - x^*\| + \sum_{j=N}^n \rho_j. \quad (16)$$

Since $\sum_{n=0}^{\infty} \rho_n < \infty$, inequality (16) implies that $\sum_{n=0}^{\infty} \alpha_n < \infty$, contradicting condition (ii). Hence $\delta = 0$, completing the proof of 3.

We now prove 4. Suppose $\lim_{n \rightarrow \infty} y_n = x^*$. Then

$$\begin{aligned} \epsilon_n &= \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Ts_n\| \\ &\leq \|y_{n+1} - x^*\| + \|(1 - \alpha_n)(y_n - x^*) + \alpha_n(Ts_n - x^*)\| \\ &\leq \|y_{n+1} - x^*\| + [1 + L(1 + L)]\|y_n - x^*\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

completing the proof of Theorem 1. ■

If we set $\beta_n = 0 \forall n \geq 0$ in Theorem 1 we obtain the following:

COROLLARY 1. *Suppose E , T , and $F(T)$ are as in Theorem 1 and $\{\alpha_n\}$ is a real sequence satisfying the conditions:*

- (i) $0 \leq \alpha_n \leq 1$,
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- (iii) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Suppose $\{x_n\}$ is the sequence generated from an arbitrary $x_0 \in E$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0.$$

Suppose $\{y_n\}$ is a sequence in E and define $\{\epsilon_n\} \subseteq \mathfrak{R}^+$ by

$$\epsilon_n = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Ty_n\|, \quad n \geq 0.$$

Then:

1. The sequence $\{x_n\}$ converges strongly to the fixed point x^* of T .
2. $\|y_n - x^*\| \leq [1 - \alpha_n r(p_n, x^*)]\|y_n - x^*\| + [L^3 + 4L^2 + 3(1 + L)]\alpha_n^2\|y_n - x^*\| + \epsilon_n$, where $p_n = (1 - \alpha_n)y_n + \alpha_n Ty_n$ and

$$r(p_n, x^*) = \frac{\phi(\|p_n - x^*\|)}{1 + \phi(\|p_n - x^*\|) + \|p_n - x^*\|}.$$

3. $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n \rightarrow \infty} y_n = x^*$, so that $\{x_n\}$ is almost T -stable.

4. $\lim_{n \rightarrow \infty} y_n = x^*$ implies $\lim_{n \rightarrow \infty} \epsilon_n = 0$.

Remark 1. For $T: E \rightarrow E$ a Lipschitz ϕ -strongly pseudocontractive map with $F(T) \neq \emptyset$, Theorem 1 and Corollary 1 show that the Ishikawa and the Mann iteration procedures considered in the Theorem 1 and the Corollary 1, respectively, are almost T -stable. The following example shows that the iteration procedures are not T -stable.

EXAMPLE 1. Let \mathfrak{R} denote the reals with the usual norm. Define $T: \mathfrak{R} \rightarrow \mathfrak{R}$ by $Tx = x/2$. Then it follows from Theorem 1 that the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in \mathfrak{R}$ by

$$\begin{aligned} z_n &= (1 - \beta_n)x_n + \beta_n Tx_n, & n \geq 0, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tz_n, & n \geq 0, \end{aligned}$$

converges strongly to the fixed point of T and is almost T -stable.

We now show that it is not T -stable.

Let $\{y_n\} \subseteq \mathfrak{R}$ be given by $y_n = n/(1+n)$, $n \geq 0$. Then

$$\begin{aligned} \epsilon_n &= |y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T((1 - \beta_n)y_n + \beta_n Ty_n)| \\ &= \left| \frac{n+1}{n+2} - \frac{n}{n+1} + \alpha_n \left[\frac{n}{n+1} - \frac{(2n - n\beta_n)}{4(n+1)} \right] \right| \\ &= \left| \frac{1}{(n+2)(n+1)} + \alpha_n \frac{(2n + n\beta_n)}{4(n+1)} \right| \\ &\leq \frac{1}{(n+2)(n+1)} + \frac{3n\alpha_n}{4(n+1)} \leq \frac{1}{(n+2)(n+1)} + \frac{3\alpha_n}{4}. \end{aligned}$$

Hence $\lim_{n \rightarrow \infty} \epsilon_n = 0$. However, $\lim y_n = 1 \neq 0 = \lim_{n \rightarrow \infty} x_n =$ the unique fixed point of T . Observe that

$$\begin{aligned} \epsilon_n &= \frac{1}{(n+2)(n+1)} + \frac{\alpha_n(2n + n\beta_n)}{4(n+1)} \geq \frac{\alpha_n(2n + n\beta_n)}{4(n+1)} \geq \frac{n\alpha_n}{2(n+1)} \\ &\geq \frac{\alpha_n}{4} \quad \forall n \geq 1, \end{aligned}$$

so that $\sum_{n=0}^{\infty} \epsilon_n = \infty$.

Remark 2. An operator T with domain $D(T)$ and range $R(T)$ in E is called ϕ -hemiccontractive (see, for example, [19]) if $F(T) = \{x \in D(T): Tx = x\} \neq \emptyset$ and for all $x \in D(T)$ and $x^* \in F(T)$ there exist $j(x - x^*) \in J(x - x^*)$ and a strictly increasing function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2 - \phi(\|x - x^*\|)\|x - x^*\|.$$

The example in [3] shows that the class of ϕ -strongly pseudocontractive operators with nonempty fixed point sets is a proper subset of the class of ϕ -hemiccontractive operators. It is easy to see that Theorem 1, Corollary 1, and Remark 1 easily extend to the class of ϕ -hemiccontractive operators.

THEOREM 2. Suppose E is an arbitrary real Banach space and $T: E \rightarrow E$ is a Lipschitz ϕ -strongly accretive operator. Suppose the equation $Tx = f$ has a solution for a given $f \in E$. Define $S: E \rightarrow E$ by $Sx = f + x - Tx$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are as in Theorem 1 and suppose $\{x_n\}$ is the sequence generated from an arbitrary $x_0 \in E$ by

$$\begin{aligned} z_n &= (1 - \beta_n)x_n + \beta_n Sx_n, & n \geq 0, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Sz_n, & n \geq 0. \end{aligned}$$

Suppose $\{y_n\}$ is a sequence in E and define $\{\epsilon_n\} \subseteq \mathfrak{R}^+$ by

$$\begin{aligned} \omega_n &= (1 - \beta_n)y_n + \beta_n Sy_n, & n \geq 0, \\ \epsilon_n &= \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n S\omega_n\|, & n \geq 0. \end{aligned}$$

Then

1. The sequence $\{x_n\}$ converges strongly to the solution x^* of the equation $Tx = f$.

$$\begin{aligned} 2. \quad \|y_{n+1} - x^*\| &\leq [1 - \alpha_n r(p_n, x^*)] \|y_n - x^*\| \\ &\quad + [L_*^3 + 4L_*^2 + 3(1 + L_*)] \alpha_n^2 \|y_n - x^*\| \\ &\quad + \alpha_n \beta_n L_* (1 + L_*) \|y_n - x^*\| + \epsilon_n, \end{aligned}$$

where $p_n = (1 - \alpha_n)y_n + \alpha_n T\omega_n$ and

$$r(p_n, x^*) = \frac{\phi(\|p_n - x^*\|)}{1 + \phi(\|p_n - x^*\|) + \|p_n - x^*\|}.$$

3. $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies that $\lim y_n = x^*$, so that $\{x_n\}$ is almost S -stable.

4. $\lim y_n = x^*$ implies that $\lim \epsilon_n = 0$.

Proof. It follows from inequality (4) that if $Tx = f$ has a solution, then the solution is unique. Let x^* denote the solution. Then x^* is a fixed point of S and S is Lipschitz with constant $L_* = 1 + L$. Furthermore, for all $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|,$$

so that S is ϕ -strongly pseudocontractive. The proof of 1 is just as given in Theorem 1 of [23] and the proofs of 2–4 are now essentially the same as the proofs of 2–4 in Theorem 1 and are therefore omitted.

COROLLARY 2. *Suppose E, T, f , and S are as in Theorem 2 and $Tx = f$ has a solution. Let $\{\alpha_n\}$ be as in Corollary 1. Suppose $\{x_n\}$ is the sequence generated from an arbitrary $x_0 \in E$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n, \quad n \geq 0.$$

Suppose $\{y_n\}$ is a sequence in E and define $\{\epsilon_n\} \subseteq \mathfrak{R}^+$ by

$$\epsilon_n = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Sy_n\|, \quad n \geq 0.$$

Then:

1. *The sequence $\{x_n\}$ converges strongly to the solution x^* of the equation $Tx = f$.*

$$\begin{aligned} 2. \quad \|y_{n+1} - x^*\| &\leq [1 - \alpha_n r(p_n, x^*)] \|y_n - x^*\| \\ &\quad + [L_*^3 + 4L_*^2 + 3(1 + L_*)] \\ &\quad \times \alpha_n^2 \|y_n - x^*\| + \epsilon_n \end{aligned}$$

where $p_n = (1 - \alpha_n)y_n + \alpha_n Ty_n$ and

$$r(p_n, x^*) = \frac{\Phi(\|p_n - x^*\|)}{1 + \phi(\|p_n - x^*\|) + \|p_n - x^*\|}.$$

3. $\sum_{n=0}^{\infty} \epsilon_n < \infty$ *implies that* $\lim y_n = x^*$, *so that* $\{x_n\}$ *is almost S -stable.*

4. $\lim y_n = x^*$ *implies that* $\lim \epsilon_n = 0$.

Remark 3. For $T: E \rightarrow E$ a Lipschitz ϕ -strongly accretive operator, if $Tx = f$ has a solution, Theorem 2 and Corollary 2 show that the Ishikawa-type and the Mann-type iteration methods given in Theorem 2 and Corollary 2, respectively, are *almost stable with respect to S* where $Sx = f + x - Tx$ and the fixed point of S is the unique solution of the equation $Tx = f$. The following example shows that the iteration methods are *not stable with respect to S* .

EXAMPLE 2. Let \mathfrak{R} and T be as in Example 1. Then the equation $Tx = f$ has a unique solution for any given $f \in \mathfrak{R}$. Define $S: \mathfrak{R} \rightarrow \mathfrak{R}$ by $Sx = f + x - Tx = f + x/2$, and let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in \mathfrak{R}$ by

$$\begin{aligned} z_n &= (1 - \beta_n)x_n + \beta_n Sx_n, & n \geq 0, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Sz_n, & n \geq 0. \end{aligned}$$

Then it follows from Theorem 2 that $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = f$, and it is almost S -stable.

We now prove that it is not S -stable. For $f = 0$, the proof follows exactly as in Example 1 with $y_n = n(n + 1)$. If $f \neq 0$, we may take $y_n = 1/(n + 1)$, $n \geq 0$. Then

$$\begin{aligned} \epsilon_n &= |y_{n+1} - (1 - \alpha_n)y_n - \alpha_n S((1 - \beta)y_n + \beta_n S y_n)| \\ &= \left| \frac{-1}{(n+2)(n+1)} + \alpha_n \left[\frac{1}{2(n+1)} + \frac{\beta_n}{4(n+1)} - \frac{\beta_n f}{2} - f \right] \right| \\ &\leq \frac{1}{(n+2)(n+1)} + \frac{1}{n+1} + \alpha_n \left(1 + \frac{\beta_n}{2} \right) |f| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

However, $\lim y_n = 0 \neq 2f = \lim x_n =$ the unique solution of the equation $Tx = f$.

Remark 4. The Mann and Ishikawa iteration methods in Theorem 1 and Corollary 1 are shown to be *almost T -stable* where T is a Lipschitz ϕ -strongly pseudocontractive map. Furthermore, the Mann-type and the Ishikawa-type iteration methods in Theorem 2 and Corollary 2 are shown to be *almost S -stable* where T is a Lipschitz ϕ -strongly accretive operator, $Sx = f + x - Tx$, and the unique fixed point of S is the unique solution of the equation $Tx = f$.

Examples 1 and 2 show that the iteration methods in Theorem 1 and Corollary 1 are not *T -stable* and that the iteration methods in Theorem 2 and Corollary 2 are not *S -stable*. It is certainly of interest to obtain Mann-type and Ishikawa-type iteration methods which are stable with respect to Lipschitz ϕ -strong pseudocontractions. Furthermore, it is of interest to obtain stable Mann-type and Ishikawa-type iteration methods for the iterative approximation of the solution (when it exists) of the equation $Tx = f$ when T is a Lipschitz ϕ -strongly accretive operator.

THEOREM 3. Suppose E is a real Banach space and $T: E \rightarrow E$ is a Lipschitz ϕ -strongly pseudocontractive operator. Suppose $F(T) \neq \emptyset$ and $\{\alpha_n\}$

and $\{\beta_n\}$ are as in Theorem 1. Suppose $\{u_n\}_{n=0}^\infty$ is any summable sequence in E (i.e., $\sum_{n=0}^\infty \|u_n\| < \infty$). Then the sequence $\{y_n\}$ generated from any $y_0 \in E$ by

$$y_{n+1} = (1 - \alpha)y_n + \alpha_n T((1 - \beta_n)y_n + \beta_n T y_n) + u_n, \quad n \geq 0, \quad (17)$$

converges strongly to the fixed point of T .

Proof. Let x^* denote the fixed point of T . From (17) we obtain

$$\|u_n\| = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T((1 - \beta_n)y_n + \beta_n T y_n)\|.$$

Since $\sum_{n=0}^\infty \|u_n\| < \infty$, it follows from Theorem 1 that $\lim y_n = x^*$.

Remark 5. If $\beta_n = 0 \forall n \geq 0$ in Theorem 3, $\{y_n\}$ reduces to the Mann iteration method with errors introduced in [15]. Thus, the almost T -stability of the Mann iteration method in Corollary 1 implies the strong convergence of the Mann iteration method with errors to the fixed point of T . Furthermore, it is clear from Theorem 3 and Example 1 that the strong convergence of the Mann iteration method with errors to the fixed point of T does not imply the T -stability of the original Mann iteration method. It is therefore more interesting to study the stability of the original Mann and Ishikawa iteration methods rather than studying these iteration methods with errors which appear to have questionable usefulness. The introduction of the error terms seems unmotivated, because under the hypotheses usually imposed on the error terms, all computations follow exactly as in the case of iteration methods without errors to yield results already known for the original iteration methods. There are no known examples where the original Mann and Ishikawa iteration methods behave differently from the Mann and Ishikawa iteration methods with errors. It appears the error terms only unnecessarily complicate the iteration schemes.

THEOREM 4. Suppose E , T , S , $\{\alpha_n\}$, and $\{\beta_n\}$ are as in Theorem 2. Suppose the equation $Tx = f$ has a solution, and $\{u_n\}$ is a summable sequence in E . Then the sequence $\{y_n\}$ generated from an arbitrary $y_0 \in E$ by

$$y_{n+1} = (1 - \alpha_n)y_n + \alpha_n S((1 - \beta_n)y_n + \beta_n S y_n) + u_n, \quad n \geq 0, \quad (18)$$

converges strongly to the solution of the equation $Tx = f$.

Proof. Let x^* denote the solution of the equation $Tx = f$. From (18) we obtain

$$\|u_n\| = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n S((1 - \beta_n)y_n + \beta_n S y_n)\|,$$

and since $\sum_{n=0}^\infty \|u_n\| < \infty$, it follows from Theorem 2 that $\lim y_n = x^*$.

Finally we present an example where the Mann and Ishikawa iteration methods are neither T -stable nor *almost* T -stable.

EXAMPLE 3 [11, p. 687]. Let \mathfrak{R} denote the reals with the usual norm. Let $T: \mathfrak{R} \rightarrow \mathfrak{R}$ be the identity mapping on \mathfrak{R} . Let $\{\alpha_n\}$ and $\{\beta_n\}$ be any sequences in $[0, 1]$ and let $\{x_n\}$ be the sequence generated from $x_0 \in \mathfrak{R}$, $x_0 \neq 0$ by

$$\begin{aligned} z_n &= (1 - \beta_n)x_n + \beta_n T x_n, & n \geq 0, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T z_n, & n \geq 0. \end{aligned}$$

Clearly $x_0 \in F(T)$ and $\{x_n\}$ converges strongly to x_0 .

Let $y_n = 1/(n + 1)$, $n \geq 0$. Then

$$\begin{aligned} \epsilon_n &= |y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T((1 - \beta_n)y_n + \beta_n T y_n)| \\ &= \frac{1}{(n + 1)(n + 2)}. \end{aligned}$$

Hence $\sum_{n=0}^{\infty} \epsilon_n < \infty$. However, $\lim y_n = 0 \neq \lim x_n = x_0 \in F(T)$. Thus $\{x_n\}$ is not almost T -stable, and hence not T -stable.

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