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NONLINEAR ACCRETIVE AND PSEUDO–CONTRACTIVE OPERATOR EQUATIONS IN BANACH SPACES

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ABSTRACT

Suppose E is *an arbitrary real Banach space* and K is a closed convex subset of E, $T : K \to K$ is a Lipschitz strong pseudo-contraction mapping. It is proved that the Ishikawa iteration scheme converges strongly to the unique fixed point of T. As a consequence of this result, it is proved that both the Mann and Ishikawa iteration methods converge strongly to the unique solutions of the operator equations $Tx = f$ and $x + Tx = f$ for a given $f \in E$ where $T : E \to E$ is a Lipschitz strong accretive map and Lipschitz accretive map respectively. Finally it is proved that these results also hold for the slightly more general class of Lipschitz strict hemi-contractions. Explicit error estimates are given and in several cases convergence is at least as fast as a geometric progression.

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1. INTRODUCTION

Let E be an arbitrary real Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is called a *strong pseudo-contraction* if there exists $t > 1$ such that for all $x, y \in D(T)$, and $r > 0$,

$$||x - y|| \leq ||(1 + r)(x - y) - rt(Tx - Ty)|| \tag{1}$$

If $t = 1$ in (1), then T is called a *pseudo-contraction.* The class of pseudo-contractive maps has been studied extensively by various authors (see for example [1],[2],[4],[6],[8-10],[14-18],[22],[24],[32],[33]). Interest in pseudo-contractive mappings stems mainly from their firm connection with the important class of *accretive operators* where a map U is called *accretive* if the inequality

$$||x - y|| \leq ||x - y + s(Ux - Uy)|| \tag{2}$$

holds for every $x, y \in D(U)$ and for all $s > 0$. Let I denote the identity operator, and observe that inequality (1) implies for $t = 1$, that

$$||x - y|| \leq ||x - y + r[(I - T)x - (I - T)y]|| \tag{3}$$

for all $x, y \in D(T)$ and $r > 0$, so that, it follows from inequalities (1) and (2) that T is pseudo-contractive if and only if $(I - T)$ is accretive. Consequently, the mapping theory for accretive operators is closely related to the fixed point theory for pseudo-contractive mappings.

Let E^* denote the dual space of E and let $J : E \to 2^{E^*}$ denote the normalized duality mapping of E defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\},$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. In the sequel we shall denote single-valued normalized duality mapping by j.

As a consequence of a result of Kato [20], it follows from inequality (2) that T is accretive if and only if for each $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq 0. \tag{4}$$

Furthermore, T is called *strongly accretive* if for each $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and a real number $k > 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq k||x - y||^2 \tag{5}$$

If $E = H$, a Hilbert space, then (4) and (5) are equivalent, respectively, to the *monotonicity* and *strong monotonicity* properties of T in the sense of Minty [25].

The accretive operators were introduced in 1967 by Browder [3] and Kato [20]. Interest in such mappings stems mainly from the fact that many physically significant problems can be modelled in terms of an initial value problem of the form

$$\frac{du}{dt} = -Tu, \ u(0) = u_0 \tag{6}$$

where T is either accretive or strongly accretive in an appropriate Banach space. Typical examples of how such evolution equations arise are found in models involving either the heat, or the wave or the Schrödinger equation (see for example [39]). An early fundamental result in the theory of accretive operators, due to Browder [3], states that the initial value problem (6) is solvable if T is locally Lipschitzian and accretive on E. Utilizing the existence result for (6), Browder [3] also showed that if T is *locally* Lipschitzian and accretive then T is *m-accretive*, i,e., $(I+T)(E) = E$.
We observe that if $N(T)$ denotes the kernel of T, then members of $N(T)$ are, in fact, the equilibrium points of the system (6). Consequently, considerable effort has been devoted to developing constructive techniques for the determination of the kernels of accretive operators (see for example [8],[10],[12],[14-17],[21],[27-29],[30],[33-35],[36],[37],[38]). Moreover, since a continuous accretive operator can be approximated well by a sequence of strongly accretive ones, particular attention has been devoted to the determination of the kernels of strongly accretive maps. In this regard and in Hilbert spaces, Vainberg [34] and Zarantonello [37] introduced the steepest descent approximation method

$$x_{n+1} = x_n - c_n T x_n, \quad x_0 \in H, \quad n = 0, 1, 2, ..., \tag{7}$$

and proved that if
(a) $T = I + M$ where I is the identity map of H and M is a monotone and Lipschitz map on H;
(b) $c_n = \lambda \in (0,1), \quad n = 0, 1, 2, ...,$
then the sequence $\{x_n\}$ defined iteratively by (7) converges strongly to an element of $N(T)$. This result has been extended to the class of bounded monotone operators (see for example [8],[12],[21],[29],[35]). Typical of the results obtained is the following theorem:
THEOREM*
Let H be a Hilbert space, $T : H \to H$ a bounded strongly accretive map with a nonempty kernel, $N(T)$. Then the sequence $\{x_n\}$ defined iteratively by (7) with $c_n \in \ell^2 \backslash \ell^1$ converges to an element of $N(T)$.
Various authors have extended Theorem* to more general Banach spaces. Vainberg [35, pp. 276-284] proved the convergence of (7) in L_p spaces for $1 < p < \infty$ where T is Lipschitz continuous and accretive; one of the authors [8] obtained the same result in L_p spaces, $p \geq 2$, under less restrictive conditions. Crandall and Pazy [12] proved convergence of (7) for a continuous strongly accretive operator on an arbitrary Banach space. Reich [29], and

also Liu [21] proved the convergence of (7) for an arbitrary strongly accretive operator on uniformly smooth Banach spaces. We remark immediately that in these results in general Banach spaces, the conditions imposed on the iteration parameter c_n are not convenient in applications. For example, Crandall and Pazy [12] required that at each iteration step, c_k be determined by

$$c_k = \frac{\delta_{k+1}}{(1+\delta_{k+1})}, \text{ where } \delta_{k+1} = 2^{-n_k},$$

and n_k is the least nonnegative integer such that

$$||T(\frac{2^n}{1+2^n}x_k - \frac{1}{1+2^n}Tx_k) - Tx_k|| \leq \exp\{-(\delta_1 + \delta_2 + ... + \delta_k + 1)\}.$$

In [29], Reich imposed the assumption that $\sum_{n=0}^{\infty} c_n^2 ||Tx_n||^2 < \infty$. Clearly, these conditions cause computational difficulties and limit the applicability of the Theorems.

Recently, the following two iteration methods have been studied by various authors for approximating solutions of nonlinear operator equations in Banach spaces.

(a) The Ishikawa Iteration Process (see for example [19],[31]) is defined as follows: For K a convex subset of a Banach space E and T a mapping of K into itself, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by $x_0 \in K$

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \ n \geq 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \ n \geq 0 \end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences satisfying the following conditions: (i) $0 \leq \alpha_n \leq \beta_n < 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$, and (iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

(b) The Mann iteration process (see for example [23],[31]) is defined as follows: With K and T as in (a) the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by $x_0 \in K$

$$x_{n+1} = (1-c_n)x_n + c_n T x_n, \ n \geq 0$$

where (i) $0 \leq c_n < 1$, (ii) $\lim_{n\to\infty} c_n = 0$, and (iii) $\sum_{n=0}^{\infty} c_n = \infty$. In some applications, condition (iii) is replaced by $\sum_{n=0}^{\infty} c_n(1-c_n) = \infty$.

The iteration processes (a) and (b) have been studied extensively by various authors and have been successfully employed to approximate fixed points of several nonlinear mappings (when these mappings are already known to have fixed points) and to approximate solutions of several nonlinear operator equations in Banach spaces (see for example [1],[5],[6-11],[14-17],[21-23],[31-33]). Moreover, it is well known that even though the two processes are similar, they may exhibit different behaviors for different classes of nonlinear mappings (see for example [31] for a detailed comparison of the two processes).

In [6], one of the authors proved the following theorem in which the iteration parameters are easily evaluated.

THEOREM C (Chidume, [6])

Suppose K is a nonempty closed convex and bounded subset of L_p, $p \geq 2$ and $T : K \to K$ is a Lipschitz strong pseudo-contraction. Suppose $\{c_n\}$ is a real sequence satisfying the following conditions:

(i) $0 < c_n < 1$ for all $n \geq 0$; (ii) $\sum_{n=0}^{\infty} c_n = \infty$; and (iii) $\sum_{n=0}^{\infty} c_n^2 < \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined iteratively by $x_0 \in K$,

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n, \quad n \geq 0,$$

converges strongly to the unique fixed point of T.

Since the publication of Theorem C, several authors have generalized it in various directions (see for example [9-11],[14-17],[22],[32],[33]). Most of these generalizations, however, involve conditions on the iteration parameters which still depend on special geometric properties of the underlying Banach spaces and therefore are not convenient in applications. For example the following theorems have recently been proved as generalizations of Theorem C:

THEOREM D (Deng, [15])

Suppose $q > 1$ and K is a closed convex and bounded subset of a q-uniformly smooth Banach space E. Suppose $T : K \to K$ is a Lipschitz strong pseudo-contraction with Lipschitz constant L, and the sequence $\{x_n\}$ is defined by $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \quad n \geq 0$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following conditions:

(i) $0 \leq \alpha_n^{s-1} \leq 2^{-1}q(k - L\beta_n - L^2\beta_n)(bL^q + h)^{-1}, \quad n \geq 0$

(ii) $0 \leq \beta_n^{s-1} \leq \min\{\frac{k}{2(L+L^2)}, \frac{qk}{(bL^q+h)}\}, \quad n \geq 0$, (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, where k is the constant appearing in the definition of a strong pseudo-contraction, b is the constant appearing in an inequality which characterizes q-uniformly smooth Banach spaces, $h = \max\{1, q(q-1)/2\}$, and $s = \min\{2, q\}$.

Then the sequence $\{x_n\}$ converges strongly to the fixed point of T

THEOREM TX (Tan and Xu, [33])

Let C be a nonempty bounded closed convex subset of a real q-uniformly smooth Banach space E with $1 < q \leq 2$ and $T : C \to C$ be a Lipschitz strongly pseudo-contractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence in $(0, 1)$ satisfying:

(i) $0 < \alpha_n \leq s_q, \quad n \geq 0$

where s_q is the (smaller) solution of the equation

$$f(s) := q(q-1)(1-k)s - (1 + d_q L^q)s^{q-1} + \frac{1}{2}qk = 0, \quad s > 0$$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$

Then for any given $x_0 \in C$*, the iteration method generated from* x_0 *by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n$$

converges strongly to the fixed point of T*. Moreover, if* $\alpha_n = s_q$ *for all* $n \geq 0$*, then*

$$||x_n - x^*|| \leq \rho^{\frac{n}{q}} ||x_0 - x^*||, \quad \text{where } \rho = (1 - \frac{1}{2} q k s_q) \in (0,1).$$

Convergence theorems similar to Theorems D and TX have also been proved for the iterative approximation of solutions of the equation $Tx = f, \quad f \in E$ where $T : E \to E$ is a Lipschitz strongly accretive operator (see [15],[33]).

Recently, Liwel Liu proved, for the Mann iteration process, the following interesting theorem in general Banach spaces and in which the iteration parameter is independent of the geometry of the underlying Banach spaces.

THEOREM L (Liwel Liu, [22])

Suppose E *is a real Banach space,* K *is a nonempty closed convex and bounded subset of* E*, and* $T : K \to K$ *is a Lipschitz strong pseudo-contraction. Let* $\{c_n\}$ *be a real sequence satisfying the following conditions: (i)* $0 \leq c_n < 1, \quad n \geq 0$*; (ii)* $\lim_{n\to\infty} c_n = 0$*; and (iii)* $\sum_{n=1}^{\infty} c_n = \infty$*. Then the sequence* $\{x_n\}$ *defined iteratively by*

$$x_{n+1} = (1-c_n)x_n + c_n T x_n, \quad n \geq 0,$$

converges strongly to the fixed point of T*.*

It is our purpose in this paper *to complement and extend Theorem L by first extending the theorem to the Ishikawa iteration process.* Our theorem will include Theorem L as a special case. Then, as a consequence of our theorem, we shall prove, *in arbitrary real Banach spaces* E and *without any geometric restrictions on the iteration parameters whatsoever*, that both the Mann and Ishikawa iteration schemes converge strongly to the unique solution of the operator equation $Tx = f$, for a given $f \in E$, where $T : E \to E$ is Lipschitzian and strongly accretive. We shall also prove convergence theorems for the operator equation $x + Tx = f$ when $T : E \to E$ is Lipschitzian and accretive. Finally, we shall prove that the above results also hold for the slightly more general class of Lipschitz *strictly hemi-contractive* maps. In all cases we shall give explicit error estimates. Our theorems generalize most of the results that have appeared recently. In particular, the results of [6],[8],[10],[14-17],[22],[33], and a host of others will be special cases of our theorems.

2. MAIN RESULTS

In the sequel $L \geq 1$ will denote the Lipschitz constant of T.

2.1. Convergence Theorems for Lipschitz Strong Pseudo-contractions

In this section $M := 3L^2 + L + 2$. We now prove the following Theorems.

THEOREM 1

Suppose E is an arbitrary real Banach space and K is a closed convex subset of E. Let $T : K \to K$ be a Lipschitz strong pseudo-contraction mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying the following conditions:

(i) $0 \leq \beta_n, \alpha_n < 1, \quad n \geq 0$

(ii) $\lim_{n\to\infty} \alpha_n = 0; \lim_{n\to\infty} \beta_n = 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty.$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by

$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0 \tag{8}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 0 \tag{9}$$

converges strongly to the fixed point of T.

PROOF The existence of a unique fixed point follows from Corollary 1 of Deimling [13]. Since T is strongly pseudo-contractive, then for all $x, y \in K$, there exist $j(x-y) \in J(x-y)$ and a constant $t > 1$ such that

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \frac{(t-1)}{t}||x-y||^2. \tag{10}$$

Set $\frac{(t-1)}{t} = k$. Then from inequality (10) we obtain

$$\langle (I-T-kI)x - (I-T-kI)y, j(x-y)\rangle \geq 0,$$

and it follows from Kato [20] that

$$||x-y|| \leq ||x - y + r[(I-T-kI)x - (I-T-kI)y||, \tag{11}$$

for all $x, y \in K$ and $r > 0$. From (9) we obtain

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T y_n \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I-T-kI)x_{n+1} - (1-k)\alpha_n x_n \\ &\quad +(2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n) \end{aligned} \tag{12}$$

Observe that

$$x^* = (1+\alpha_n)x^* + \alpha_n(I-T-kI)x^* - (1-k)\alpha_n x^*, \tag{13}$$

so that

$$\begin{aligned} x_n - x^* &= (1+\alpha_n)(x_{n+1} - x^*) + \alpha_n[(I-T-kI)x_{n+1} - (I-T-kI)x^*] \\ &\quad -(1-k)\alpha_n(x_n - x^*) + (2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n) \end{aligned}$$

Hence

$$\begin{aligned}||x_n - x^*|| &\geq (1+\alpha_n)||(x_{n+1}-x^*) + \frac{\alpha_n}{(1+\alpha_n)}[(I-T-kI)x_{n+1} - (I-T-kI)x^*]|| \\ &\quad -(1-k)\alpha_n||x_n - x^*|| - (2-k)\alpha_n^2||x_n - Ty_n|| - \alpha_n||Tx_{n+1} - Ty_n|| \\ &\geq (1+\alpha_n)||(x_{n+1}-x^*)|| - (1-k)\alpha_n||x_n - x^*|| - (2-k)\alpha_n^2||x_n - Ty_n|| \\ &\quad -\alpha_n||Tx_{n+1} - Ty_n||\end{aligned}$$

Furthermore, we have the following estimates:

$$||x_{n+1}-x^*|| \leq \frac{[1+(1-k)\alpha_n]}{(1+\alpha_n)}||x_n - x^*|| + (2-k)\alpha_n^2||x_n - Ty_n|| + \alpha_n||Tx_{n+1} - Ty_n|| \quad (14)$$

$$||y_n - x^*|| = ||(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)|| \leq [1+(L-1)\beta_n]||x_n - x^*|| \leq L||x_n - x^*||$$

$$||x_n - Ty_n|| \leq ||x_n - x^*|| + L||y_n - x^*|| \leq [1+L^2]||x_n - x^*||, \quad (15)$$

$$\begin{aligned}||Tx_{n+1} - Ty_n|| \leq L||x_{n+1} - y_n|| &= L||(1-\alpha_n)(x_n - y_n) + \alpha_n(Ty_n - y_n)|| \\ &\leq L(1-\alpha_n)\beta_n||x_n - Tx_n|| + \alpha_n L(1+L)||x_n - x^*|| \\ &\leq [L(1+L)(1-\alpha_n)\beta_n + L(1+L)\alpha_n]||x_n - x^*|| \\ &= K_n||x_n - x^*|| \quad (16)\end{aligned}$$

where $K_n := L(1+L)(1-\alpha_n)\beta_n + L(1+L)\alpha_n$. Using (15) and (16) in (14) we obtain the following estimates:

$$\begin{aligned}||x_{n+1} - x^*|| &\leq \frac{[1+(1-k)\alpha_n]}{(1+\alpha_n)}||x_n - x^*|| + [(2-k)(1+L^2)]\alpha_n^2||x_n - x^*|| + K_n\alpha_n||x_n - x^*|| \\ &\leq [1+(1-k)\alpha_n][1-\alpha_n+\alpha_n^2]||x_n - x^*|| + [2(1+L^2)+L(1+L)]\alpha_n^2||x_n - x^*|| \\ &\quad +L(1+L)(1-\alpha_n)\alpha_n\beta_n||x_n - x^*|| \\ &\leq [1-k\alpha_n]||x_n - x^*|| + \alpha_n||x_n - x^*||[\{2(1+L^2)+L(1+L)\}\alpha_n + L(1+L)\beta_n] \\ &\leq [1-k\alpha_n]||x_n - x^*|| + M(\alpha_n+\beta_n)\alpha_n||x_n - x^*|| \\ &= [1-k\alpha_n + M(\alpha_n+\beta_n)\alpha_n]||x_n - x^*|| \quad (17)\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0,\ \lim_{n\to\infty}\beta_n = 0$ there exists an integer $N_0 \geq 0$ such that

$$M(\alpha_n+\beta_n) \leq k(1-k), \quad \forall\ n \geq N_0.$$

Thus

$$\begin{aligned}||x_{n+1} - x^*|| &\leq [1-k^2\alpha_n]||x_n - x^*||, \quad \forall\ n \geq N_0 \\ &\leq ||x_{N_0} - x^*||\exp(-k^2\sum_{j=N_0}^{n}\alpha_j) \to 0 \text{ as } n\to\infty, \text{ by (iii)},\end{aligned}$$

completing proof of Theorem 1.

COROLLARY 1 *Suppose E, K, and T are as in Theorem 1. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the following conditions:*
(i) $0 \leq \alpha_n < 1, \quad n \geq 0$
(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$
(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0$$

converges strongly to the fixed point of T.

PROOF
Obvious from Theorem 1.

COROLLARY 2 *Suppose E, K and T are as in Theorem 1. Let $\lambda = \frac{k}{4M}$, $T_\lambda := (1-\lambda)I + \lambda T$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by $x_{n+1} = T_\lambda x_n, \quad n \geq 0$ converges strongly to the fixed point x^* of T and moreover,*

$$||x_n - x^*|| \leq \rho^n ||x_0 - x^*||,$$

where $\rho = (1 - \frac{k^2}{8M}) \in (0,1)$.

PROOF Set $\alpha_n = \beta_n = \lambda$ for each n in inequality (17) to obtain

$$||x_{n+1} - x^*|| \leq [1 - \lambda k + 2M\lambda^2]||x_n - x^*|| = \rho||x_n - x^*||.$$

Iteration of this inequality now yields the desired result.

2.2. Convergence Theorems for Lipschitz Strict Hemi-contractions

Let E be a real Banach space. A mapping T with domain D(T) and range R(T) in E is called a *strict hemi-contraction* (see for example [11]) if
$F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset$ and for all $x \in D(T)$, $x^* \in F(T)$, and $r > 0$ there exists $t > 1$ such that

$$||x - x^*|| \leq ||(1+r)(x - x^*) - rt(Tx - x^*)||$$

Remark 1 Every strongly pseudo-contractive mapping with a nonempty fixed point set is strictly hemi-contractive. An example of a Lipschitz strictly hemi-contractive mapping which is not strongly pseudo-contractive is given in [11]. Consequently, the class of Lipschitz strong pseudo-contractions with nonempty fixed point sets is a *proper* subclass of the class of Lipschitz strictly hemi-contractions. It is also shown in [11] that T is strictly hemi-contractive if and only if

$$\langle x - Tx, j(x - x^*)\rangle \geq (\frac{t-1}{t})||x - x^*||^2,$$

for all $x \in D(T)$ and $x^* \in F(T)$. With M as in section 2.1 and using the method of proof of Theorem 1, the following Theorem is easily proved.

THEOREM 2

Suppose E is an arbitrary real Banach space and K is a closed convex subset of E. Let $T : K \to K$ be a Lipschitz strict hemi-contraction mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying the following conditions:

(i) $0 \leq \beta_n,\ \alpha_n < 1, \quad n \geq 0$

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0, \ \lim\limits_{n\to\infty} \beta_n = 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty.$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by (8) and (9) converges strongly to the fixed point of T.

Remark 2 It follows from Remark 1 that Theorem 1 is in fact a corollary of Theorem 2.

COROLLARY 3 *Suppose E, K, and T are as in Theorem 2. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the following conditions:*

(i) $0 \leq \alpha_n < 1, \quad n \geq 0$

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty.$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0$$

converges strongly to the fixed point of T.

PROOF Obvious from Theorem 2.

COROLLARY 4 *Suppose E, K and T are as in Theorem 2. Let $\lambda = \frac{k}{4M}$, $T_\lambda := (1-\lambda)I + \lambda T$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in K$ by $x_{n+1} = T_\lambda x_n, \quad n \geq 0$ converges strongly to the fixed point x^* of T and moreover,*

$$||x_n - x^*|| \leq \rho^n ||x_0 - x^*||,$$

where $\rho = (1 - \frac{k^2}{8M}) \in (0, 1)$.

PROOF Follows as in the proof of Corollary 2.

2.3. Convergence Theorems for the Equation $Tx = f$

In this section $L_* := 1 + L$ and $M^* := 3L_*^2 + L_* + 2$.

THEOREM 3

Suppose E is an arbitrary real Banach space and $T : E \to E$ is a Lipschitz strong accretive operator. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying:

(i) $0 \le \beta_n,\ \alpha_n < 1, \quad n \ge 0$

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \lim\limits_{n\to\infty} \beta_n = 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from any $x_0 \in E$ by

$$y_n = (1 - \beta_n)x_n + \beta_n(f + (I - T)x_n), \quad n \ge 0 \tag{18}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + (I - T)y_n), \quad n \ge 0 \tag{19}$$

converges strongly to the unique solution of the equation $Tx = f, \quad f \in E$.

PROOF The existence of a solution to $Tx = f$ follows from Browder [3] and the uniqueness follows from the strong accretivity condition of T. Define $S : E \to E$ by $Sx = f + (I - T)x$. Let x^* denote the solution. Then x^* is a fixed point of S and S is Lipschitz with constant $L_* = 1 + L$. Furthermore,

$$\langle (I - S)x - (I - S)y, j(x - y)\rangle \ge k||x - y||^2, \quad \forall x, y \in E,$$

so that S is strongly pseudo-contractive. The rest of the argument is now essentially the same as in the proof Theorem 1 and is therefore omitted.

COROLLARY 5 *Suppose E and T are as in Theorem 3. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the following conditions:*

(i) $0 \le \alpha_n < 1, \quad n \ge 0$

(ii) $\lim\limits_{n=0}^{\infty} \alpha_n = 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the solution of the equation $Tx = f$, $f \in E$.

PROOF Obvious from Theorem 3.

COROLLARY 6 *Suppose E and T are as in Theorem 3. Let $\lambda = \frac{k}{4M^*}$, and $S_\lambda = (1-\lambda)I + \lambda S$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from any $x_0 \in E$ by $x_{n+1} = S_\lambda x_n, \quad n \geq 0$ converges strongly to the solution x^* of the equation $Tx = f, \quad f \in E$. Moreover,*

$$||x_{n+1} - x^*|| \leq \rho^n ||x_0 - x^*||,$$

where

$$\rho = \left(1 - \frac{k^2}{8M^*}\right) \in (0,1).$$

PROOF Follows as in the proof of Corollary 2.

2.4. Convergence Theorems for the Equation $x + Tx = f$

THEOREM 4

Suppose E is an arbitrary real Banach space and $T : E \to E$ is a Lipschitz accretive operator. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying:

(i) $0 \leq \beta_n, \ \alpha_n < 1, \quad n \geq 0$

(ii) $\lim_{n\to\infty} \alpha_n = 0, \ \ \lim_{n\to\infty} \beta_n = 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in E$ by

$$y_n = (1-\beta_n)x_n + \beta_n(f - Tx_n), \quad n \geq 0 \tag{20}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f - Ty_n), \quad n \geq 0 \tag{21}$$

converges strongly to the solution of the equation $x + Tx = f$.

PROOF The existence of a solution to the equation follows from Browder [3] and the uniqueness follows from the accretivity condition of T. Define $S : E \to E$ by $Sx = f - Tx$. Let x^* denote the solution. Then x^* is a fixed point of S and S is Lipschitz with the same constant L as T. Furthermore,

$$\langle Sx - Sy, j(x-y)\rangle = -\langle Tx - Ty, j(x-y)\rangle \leq 0,$$

for all $x, y \in E$, so that (-S) is accretive. Thus

$$||x - y|| \leq ||x - y - r(Sx - Sy)|| \tag{22}$$

for all $x, y \in E$ and $r > 0$.
From (21) we obtain

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n S y_n \\ &= (1+\alpha_n)x_{n+1} - \alpha_n S x_{n+1} + \alpha_n^2(x_n - Sy_n) + \alpha_n(Sx_{n+1} - Sy_n) \end{aligned} \tag{23}$$

Also

$$x^* = (1+\alpha_n)x^* - \alpha_n Sx^*, \tag{24}$$

so that

$$x_n - x^* = (1+\alpha_n)(x_{n+1} - x^*) - \alpha_n(Sx_{n+1} - Sx^*) + \alpha_n^2(x_n - Sy_n) + \alpha_n(Sx_{n+1} - Sy_n).$$

Hence

$$\begin{aligned}
||x_n - x^*|| &\geq ||(1+\alpha_n)(x_{n+1} - x^*) - \alpha_n(Sx_{n+1} - Sx^*)|| \\
&\quad -\alpha_n^2||(x_n - Sy_n)|| - \alpha_n||(Sx_{n+1} - Sy_n)|| \\
&\geq (1+\alpha_n)||x_{n+1} - x^*|| - \alpha_n^2||(x_n - Sy_n)|| - \alpha_n||(Sx_{n+1} - Sy_n)||
\end{aligned}$$

Using (15) and (16) we obtain the following estimates:

$$\begin{aligned}
||x_{n+1} - x^*|| &\leq \frac{1}{(1+\alpha_n)}||x_n - x^*|| + \alpha_n^2||(x_n - Sy_n)|| \\
&\quad +\alpha_n||(Sx_{n+1} - Sy_n)|| \\
&\leq [1 - \alpha_n + \alpha_n^2]||x_n - x^*|| + \alpha_n^2||(x_n - Sy_n)|| \\
&\quad +\alpha_n||(Sx_{n+1} - Sy_n)|| \\
&\leq [1 - \alpha_n + \alpha_n^2]||x_n - x^*|| + \alpha_n^2(1+L^2)||x_n - x^*|| + \alpha_n K_n||x_n - x^*|| \\
&\leq [1 - \alpha_n]||x_n - x^*|| + K_1\alpha_n^2||x_n - x^*|| \\
&\quad +[L(1+L)(1-\alpha_n)\alpha_n\beta_n + L(1+L)\alpha_n^2]||x_n - x^*|| \\
&\leq (1-\alpha_n)||x_n - x^*|| \\
&\quad +\alpha_n[K_1\alpha_n + L(1+L)\beta_n + L(1+L)\alpha_n]||x_n - x^*||
\end{aligned}$$

where $K_1 := 2 + L^2$. Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$, then for arbitrary $k \in (0,1)$ there exists an integer $N_0 \geq 0$ such that $K_1\alpha_n + L(1+L)\beta_n + L(1+L)\alpha_n \leq k \quad \forall \; n \geq N_0$. Thus

$$\begin{aligned}
||x_{n+1} - x^*|| &\leq [1 - \alpha_n(1-k)]||x_n - x^*||, \quad \forall \; n \geq N_0 \\
&\leq ||x_{N_0} - x^*||\exp(-(1-k)\sum_{j=N_0}^{n}\alpha_j) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

completing the proof of Theorem 4.

COROLLARY 7 *Suppose E and T are as in Theorem 4. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the following conditions:*

(i) $0 \leq \alpha_n < 1, \quad n \geq 0$

(ii) $\lim_{n\to\infty}\alpha_n = 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then the sequence $\{x_n\}_{n=0}^{\infty}$ *generated from arbitrary* $x_0 \in E$ *by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f - Tx_n), \quad n \geq 0$$

converges strongly to the solution of the equation $x + Tx = f$.

PROOF Obvious from Theorem 4.

COROLLARY 8 *Suppose* E *and* T *are as in Theorem 4. Let* $\lambda = \frac{1}{4M}$ *and* $S_\lambda = (1 - \lambda)I + \lambda S$. *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated from any* $x_0 \in E$ *by* $x_{n+1} = S_\lambda x_n, \quad n \geq 0$, *converges strongly to the solution,* x^*, *of the equation* $x + Tx = f$. *Moreover,*

$$||x_{n+1} - x^*|| \leq \rho^n ||x_{N_0} - x^*||,$$

where $\rho = (1 - \frac{1}{8M}) \in (0, 1)$.

PROOF Follows as in the proof of Corollary 2.

Remark 3. Theorem 1 is a significant generalization of Theorems D (Deng [15]) and TX (Tan and Xu [33]) in the following sense:

1. Theorem 1 holds in *arbitrary real Banach spaces* whereas Theorems D and TX have been proved in the restricted $q-$ uniformly smooth Banach spaces.

2. In Theorem 1, unlike in Theorems D and TX, the *boundedness* of the subset K is not required.

3. The restrictions on the iteration parameters α_n, β_n in Theorems D and TX (see conditions (i) and (ii) in Theorem D and condition (ii) in Theorem TX) are most undesirable. In our Theorems, these parameters are not dependent on either the geometry of the underlying Banach spaces or on the Lipschitz constant of the operator. In fact, they can be chosen at the beginning of the iteration process. For example, in Theorem 1, a prototype for them is $\alpha_n = \beta_n = \frac{1}{n+1}$, $n = 1, 2, ...$.

Remark 4 It is now easy to see that Theorem 1 and Corollary 1 are significant generalizations of Theorem C, Theorem D, Theorem TX, and a host of other theorems (see for example Theorems 1 and 2 of ([8],[14],[16],[17]), Theorems 1-4 of [15], Theorem 2 of [32], Theorems 3.1, 3.2, 4.1, and 4.2 of [33]) to *arbitrary real Banach spaces* E and *without any dependence* of the iteration parameters α_n, β_n on the geometric structure of E. Moreover, for the special choices of the iteration parameters, Corollaries 2, 3, 6 and

8 yield convergence rate which is at least as fast as a geometric progression, better than that obtainable from any of Theorems C, D and TX. Theorem 2 extends Theorem 1 to the more general class of Lipschitz strict hemi-contractions.

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