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ITERATIVE SOLUTIONS
OF NONLINEAR ACCRETIVE OPERATOR EQUATIONS
IN ARBITRARY BANACH SPACES

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ABSTRACT

Suppose $E$ is an arbitrary real Banach space and $K$ is a nonempty closed convex and bounded subset of $E$. Suppose $T : K \to K$ is a uniformly continuous strong pseudo-contraction. It is proved that the Mann and the Ishikawa iteration methods converge strongly to the unique fixed point of $T$. Furthermore, our results also hold for the slightly more general class of strictly hemicontractive maps. Related results deal with the iterative approximation of solutions of accretive operator equations in arbitrary real Banach spaces.

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1. INTRODUCTION

Suppose $E$ is an arbitrary real Banach space. We denote by $J$ the normalized duality mapping from $E$ into $2^{E^*}$ given by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = ||x||^2 = ||f||^2 \}$$

where $E^*$ denotes the dual space of $E$ and $\langle . , . \rangle$ denotes the generalized duality pairing. It is well known that if $E^*$ is strictly convex then $J$ is single-valued. In the sequel we shall denote single-valued normalized duality mapping by $j$.

An operator $T$ with domain $D(T)$ and range $R(T)$ in $E$ is called a strong pseudocontraction if for all $x, y \in D(T)$ there exist $j(x - y) \in J(x - y)$ and $t > 1$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \frac{1}{t} ||x - y||^2.$$  \hspace{1cm} (1)

The class of strong pseudocontractions has been studied extensively by several authors (see for example [2],[3-5],[6-10],[12-16],[22-28]). Interest in such mappings stems mainly from their firm connection with the important class of strongly accretive operators where an operator $U$ is called strongly accretive if for all $x, y \in D(U)$ there exist $j(x - y) \in J(x - y)$ and a constant $k > 0$ such that

$$\langle Ux - Uy, j(x - y) \rangle \geq k||x - y||^2.$$  \hspace{1cm} (2)

If $I$ denotes the identity operator on $E$, then it follows from inequalities (1) and (2) that $T$ is strongly pseudocontractive if and only if $(I-T)$ is strongly accretive. Thus the mapping theory for strongly accretive operators is closely related to the fixed point theory of strongly pseudocontractive operators. If $k = 0$ in (2) then $U$ is called accretive. If $U$ is accretive and $(I + rU)(D(U)) = E$ for all $r > 0$ then $U$ is called $m$-accretive.

It is well known (see for example Deimling [11]) that if $T : E \to E$ is continuous and strongly pseudocontractive, then $T$ has a unique fixed point. Furthermore, if $T : E \to E$ is continuous and strongly accretive, then $T$ is surjective, i.e., for a given $f \in E$, the equation

$$Tx = f$$  \hspace{1cm} (3)
has a unique solution. Martin [20] has also proved that if $T : E \rightarrow E$ is continuous and accretive, then $T$ is $m$-accretive so that the equation

$$x + Tx = f$$  \hspace{1cm} (4)$$

has a unique solution for any $f \in E$.

Recently, several authors have applied the Mann iteration method [19] and the Ishikawa iteration method [17] to approximate fixed points of strong pseudocontractions with nonempty fixed-point sets and solutions (when they exist) of equations (3) and (4) (see for example [2], [6-10], [12-15], [22-28]). A fundamental application of the Mann iteration method to the iterative approximation of fixed points of strongly pseudocontractive maps is the following theorem.

**THEOREM C1 ([6], p. 285.)**

Suppose $K$ is a nonempty closed convex and bounded subset of $L_p$, $p \geq 2$ and $T : K \rightarrow K$ is a Lipschitz strong pseudocontraction. Suppose $\{c_n\}$ is a real sequence satisfying the following conditions: (i) $0 < c_n < 1$ for all $n \geq 0$, (ii) $\sum_{n=0}^{\infty} c_n = \infty$, and (iii) $\sum_{n=0}^{\infty} c_n^2 < \infty$. Then the sequence of Mann iterates $\{x_n\}^{\infty}_{n=0}$ defined for arbitrary $x_0 \in K$ by

$$x_{n+1} = (1 - c_n)x_n + c_nTx_n, \hspace{1cm} n \geq 0$$

converges strongly to the unique fixed point of $T$.

Since the publication of Theorem C1, several authors have generalized and extended it in various directions (see for example [8-10], [12-15], [22-28]).

Recently, using a fundamental result from the theory of ordinary differential equations Schu [26] proved the following generalization of Theorem C1.

**THEOREM S ([26], p.70)**

Suppose $E$ is a real smooth Banach space and $K$ is a nonempty closed convex and bounded subset of $E$. Suppose $T : K \rightarrow K$ is a uniformly continuous strongly pseudo-contractive map and $\alpha (x, y)$ is a nondecreasing function on $E$ with $\alpha (x, x) = 0$ for all $x \in K$. Then $T$ has a unique fixed point.

Since $T$ is uniformly continuous, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|z - z'\| < \delta$ implies $\|Tz - Tz'\| < \epsilon$ for all $z, z' \in K$. Let $h : K \rightarrow K$, $h(x) = x + \alpha (x, x')$ for $x, x' \in K$. Then $h$ is a strongly accretive map, and it follows from the above that $h$ has a unique fixed point $x_0$. Since $T$ is accretive, $T$ has a unique fixed point.
contractive map and suppose for some fixed positive integer \( N_0 \), \( \{\alpha_n\} \subseteq (0,1)^{N_0} \) satisfies the conditions: (i) \( \sum_{n=N_0}^{\infty} \alpha_n = \infty \), (ii) \( \lim_{n \to \infty} \alpha_n = 0 \). Then the sequence of the Mann iterates \( \{x_n\} \) defined for arbitrary \( x_0 \in K \) by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad (n \geq N_0)
\]

converges strongly to the unique fixed point of \( T \).

Theorem S extends Theorem C1 to the more general real smooth Banach Spaces and to the more general uniformly continuous maps.

Recently, one of the authors [10], using the method of the proof of Theorem S extended this theorem to the Ishikawa iteration method. More precisely, the following theorem has been proved.

**THEOREM C2 ([10], p. 1827)**

Suppose \( E \) is a real smooth Banach space and \( K \) a nonempty closed convex and bounded subset of \( E \). Suppose \( T : K \to K \) is a uniformly continuous strongly pseudocontractive map. For some fixed positive integer \( N_0 \), suppose \( \{\alpha_n\} \subseteq (0,1)^{N_0} \), \( \{\beta_n\} \subseteq (0,1)^{N_0} \) are such that (i) \( 0 < \alpha_n < \beta_n < 1 \ \forall n \geq N_0 \), (ii) \( \lim_{n \to \infty} \beta_n = 0 \), and (iii) \( \sum_{n=N_0}^{\infty} \alpha_n = \infty \). Then the sequence of the Ishikawa iterates \( \{x_n\} \) generated from an arbitrary \( x_0 \in K \) by

\[
y_n = (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq N_0
\]

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \quad n \geq N_0
\]

converges strongly to the unique fixed point of \( T \).

**Remark 1** Theorem C2 does not include Theorem S as a special case because of condition (i) of Theorem C2. It is certainly of interest to obtain a theorem similar to Theorem C2 which does include Theorem S as a special case.

Applications of Theorem S and C2 to iterative approximation of solutions of equations (3) and (4) have also been given in [10]. Theorems S and C2 are significant generalizations of important known results (see for example [6-9],[12-15],[23], [27]).
It is our purpose in this paper to extend Theorems S and C2 from real smooth Banach spaces to arbitrary real Banach spaces. Our proofs are independent of any result from the theory of ordinary differential equations. Moreover, our theorems also hold for the slightly more general class of strictly hemicontractive maps. As applications of our theorems, we prove that both the Mann and the Ishikawa iteration methods converge strongly to the solution of (3) when $E$ is an arbitrary real Banach space and $T : E \to E$ is uniformly continuous and strongly accretive and the range of $(I - T)$ is bounded. We also prove similar convergence result for the operator equation (4) when $T : E \to E$ is a uniformly continuous accretive operator with a bounded range.

We shall need the following result:

**LEMMA W** ([28], p.729)

Suppose $\{\rho_n\}_{n=0}^\infty$ is a nonnegative sequence satisfying the following inequality:

$$\rho_{n+1} \leq (1 - \delta_n)\rho_n + \sigma_n, \quad n \geq 0$$

with $\delta_n \in [0, 1], \sum_{n=0}^\infty \delta_n = \infty$, and $\sigma_n = o(\delta_n)$. Then $\lim_{n \to \infty} \rho_n = 0$.

**2. MAIN RESULTS**

**2.1 Convergence Theorems for Uniformly Continuous Strong Pseudocontractions**

**THEOREM 1**

Suppose $E$ is an arbitrary real Banach space and $K$ is a nonempty closed convex bounded subset of $E$. Suppose $T : K \to K$ is a uniformly continuous strong pseudocontraction and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following conditions:

(i) $0 \leq \alpha_n, \beta_n \leq 1, \quad n \geq 0$
(ii) $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0$, and (iii) $\sum_{n=0}^\infty \alpha_n = \infty$. Then the sequence $\{x_n\}_{n=0}^\infty$ generated from an arbitrary $x_0 \in K$ by

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \geq 0$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n,$$

(5)
converges strongly to the fixed point of $T$.

**Proof** The existence of a unique fixed point of $T$ follows from Corollary 1 of Deimling [11]. Since $T$ is strongly pseudocontractive, it follows from (1) that for all $x, y \in K$ there exist $j(x - y) \in J(x - y)$ and a constant $t > 1$ such that

$$
\langle (I - T)x - (I - T)y, j(x - y) \rangle \geq \frac{(t - 1)}{t} ||x - y||^2. \tag{6}
$$

Set $\frac{(t - 1)}{t} = k$. Then from inequality (6) we obtain

$$
\langle (I - T - kI)x - (I - T - kI)y, j(x - y) \rangle \geq 0,
$$

and it follows from Kato [18] that

$$
||x - y|| \leq ||x - y + r((I - T - kI)x - (I - T - kI)y)||, \tag{7}
$$

for all $x, y \in K$ and $r > 0$. From (5) we obtain

$$
x_n = x_{n+1} + \alpha_n x_n - \alpha_n Ty_n
= (1 + \alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} - (1 - k)\alpha_n x_n
+ (2 - k)\alpha_n^2(x_n - Ty_n) + \alpha_n(T x_{n+1} - Ty_n) \tag{8}
$$

Observe that

$$
x^* = (1 + \alpha_n)x^* + \alpha_n(I - T - kI)x^* - (1 - k)\alpha_n x^*, \tag{9}
$$

so that

$$
x_n - x^* = (1 + \alpha_n)(x_{n+1} - x^*) + \alpha_n[(I - T - kI)x_{n+1} - (I - T - kI)x^*]
-(1 - k)\alpha_n(x_n - x^*) + (2 - k)\alpha_n^2(x_n - Ty_n) + \alpha_n(T x_{n+1} - Ty_n).
$$

Hence

$$
||x_n - x^*|| \geq (1 + \alpha_n)||x_{n+1} - x^*|| + \frac{\alpha_n}{1 + \alpha_n}[||(I - T - kI)x_{n+1} - (I - T - kI)x^*)||
-(1 - k)\alpha_n||x_n - x^*|| - (2 - k)\alpha_n^2||x_n - Ty_n|| - \alpha_n||T x_{n+1} - Ty_n||
\geq (1 + \alpha_n)||x_{n+1} - x^*|| - (1 - k)\alpha_n||x_n - x^*|| - (2 - k)\alpha_n^2||x_n - Ty_n||
-\alpha_n||T x_{n+1} - Ty_n||, \text{ (using (7)).}
$$
Thus

\[ |x_{n+1} - x^*| \leq \frac{1 + (1 - k)\alpha_n}{1 + \alpha_n} |x_n - x^*| + (2 - k)\alpha_n^2 |x_n - Ty_n| + \alpha_n |Tx_{n+1} - Ty_n| \]

\[ \leq [1 + (1 - k)\alpha_n][1 - \alpha_n + \alpha_n^2] |x_n - x^*| + (2 - k)\alpha_n^2 |x_n - Ty_n| \]

\[ + \alpha_n |Tx_{n+1} - Ty_n| \]

\[ \leq [1 - k\alpha_n + \alpha_n^2] |x_n - x^*| + (2 - k)\alpha_n^2 |x_n - Ty_n| + \alpha_n |Tx_{n+1} - Ty_n| \]

\[ \leq [1 - k\alpha_n] |x_n - x^*| + (3 - k) \text{Diam}(K)\alpha_n^2 + \alpha_n |Tx_{n+1} - Ty_n| \quad (10) \]

Observe that

\[ |x_{n+1} - y_n| = |(1 - \alpha_n)(x_n - y_n) + \alpha_n(Ty_n - y_n)| \]

\[ \leq (1 - \alpha_n)\beta_n |x_n - Tx_n| + \alpha_n |Ty_n - y_n| \]

\[ \leq [\beta_n + \alpha_n] \text{Diam}(K) \to 0 \text{ as } n \to \infty, \]

so that it follows from the uniform continuity of \( T \) that

\[ \lim_{n \to \infty} |Tx_{n+1} - Ty_n| = 0. \]

Set \( k\alpha_n = \delta_n \), and \((3 - k)\text{Diam}(K)\alpha_n^2 + \alpha_n |Tx_{n+1} - Ty_n| = \sigma_n \) in (10) to obtain

\[ |x_{n+1} - x^*| \leq [1 - \delta_n] |x_n - x^*| + \sigma_n. \]

Clearly \( \sum_{n=0}^{\infty} \delta_n = \infty \), and \( \sigma_n = o(\delta_n) \) and an application of Lemma W now implies that

\[ |x_{n+1} - x^*| \to 0 \text{ as } n \to \infty, \]

completing the proof of Theorem 1.

**COROLLARY 1**

Let \( E, K \), and \( T \) be as in Theorem 1. Let \( \{\alpha_n\} \) be a real sequence satisfying the conditions:

(i) \( 0 \leq \alpha_n \leq 1 \), (ii) \( \lim_{n \to \infty} \alpha_n = 0 \), and (iii) \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Then the sequence \( \{x_n\} \) generated from an arbitrary \( x_0 \in K \) by

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0 \]

converges strongly to the fixed point of \( T \).
PROOF Obvious from Theorem 1.

Remark 2 Theorem 1 extends Theorem S, Theorem C1 and a host of other theorems (see for example Theorems 1 and 2 of ([12],[14],[15]), Theorems 1-4 of [13], Theorem 2 of [25], Theorems 3.1, 3.2, 4.1 and 4.2 of [27]) from either real smooth or real uniformly convex Banach spaces to arbitrary real Banach spaces $E$ and without any dependence of the iteration parameters $\{\alpha_n\}$ and $\{\beta_n\}$ on the geometric structure of $E$ or on any property of the operator $T$. A prototype for $\{\alpha_n\}$ and $\{\beta_n\}$ in Theorem 1 is $\alpha_n = \beta_n = \frac{1}{1+n}$, $n \geq 0$. Moreover, the proof of Theorem 1 does not require the use of results from the theory of ordinary differential equations.

Remark 3 Suppose $E$ is an arbitrary real Banach space. A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is called strictly hemicontractive (see for example Chidume and Osilike [24]) if $F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset$ and for all $x \in D(T)$, $x^* \in F(T)$, and $r > 0$ there exists $t > 1$ such that

$$||x - x^*|| \leq ||(1 + r)(x - x^*) - rt(Tx - x^*)||.$$}

Every strongly pseudocontractive mapping with a nonempty fixed-point set is strictly hemicontractive. An example of a strictly hemicontractive mapping which is not strongly pseudocontractive is given in [24]. It is also shown in [24] that $T$ is strictly hemicontractive if and only if

$$\langle x - Tx, j(x - x^*)\rangle \geq \frac{(t - 1)}{t}||x - x^*||^2$$

for some $j(x - x^*) \in J(x - x^*)$ and for all $x \in D(T)$ and $x^* \in F(T)$. Consequently, a close examination of the proof of Theorem 1 reveals that it is easily extendible to the more general class of strictly hemicontractive maps.

2.2 Convergence Theorems for the Equation $Tx = f$

In this section $k \in (0, 1)$ is the constant appearing in the definition of strongly accretive operators.
THEOREM 2

Suppose $E$ is an arbitrary real Banach space and $T : E \to E$ is a uniformly continuous strongly accretive operator. Suppose the range of $(I - T)$ is bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Theorem 1. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in E$ by

$$y_n = (1 - \beta_n)x_n + \beta_n(f + (I - T)x_n), \quad n \geq 0$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + (I - T)y_n), \quad n \geq 0$$

converges strongly to the unique solution of the equation $Tx = f$.

PROOF  The existence of a solution to the equation follows from Deimling [11] (see also Martin [20]) and the uniqueness follows from the strong accretivity condition of $T$. Define $S : E \to E$ by $Sx = f + (I - T)x$. Let $x^*$ denote the solution. Then $x^*$ is a fixed point of $S$ and $S$ is uniformly continuous. Furthermore,

$$\langle (I - S)x - (I - S)y, j(x - y) \rangle \geq k\|x - y\|^2, \quad \forall x, y \in E,$$

so that $S$ is strongly pseudocontractive. It now follows as in the proof of Theorem 1 that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \alpha_n\|Sx_n - x^*\| + \alpha_n\|x_n - x^*\|. \quad (11)$$

Let $D := \sup\{\|Sy_n - x^*\| + \|Sx_n - x^*\|, n \geq 0\} + \|x_0 - x^*\|$. Then by a simple induction we obtain

$$\|x_n - x^*\| \leq D \quad \forall n \geq 0. \quad (12)$$

Furthermore, $\|y_n - x^*\| \leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|Sx_n - x^*\| \leq D$. Thus,

$$\|x_n - Sy_n\| \leq \|x_n - x^*\| + \|x^* - Sy_n\| \leq 2D \quad (13)$$

Also

$$\|x_{n+1} - y_n\| \leq (1 - \alpha_n)\|x_n - y_n\| + \alpha_n\|Sy_n - y_n\|$$
$$\leq (1 - \alpha_n)\beta_n\|x_n - Sx_n\| + \alpha_n\|Sy_n - x^*\| + \alpha_n\|y_n - x^*\|$$
$$\leq 2D(\beta_n + \alpha_n) \to 0 \quad \text{as} \quad n \to \infty,$$
so that the uniform continuity of \( S \) implies that
\[
\lim_{n \to \infty} ||Sx_{n+1} - Sy_n|| = 0.
\]

Using (12) and (13) in (11) we obtain
\[
||x_{n+1} - x^*|| \leq [1 - k\alpha_n]||x_n - x^*|| + (5 - 2k)D\alpha_n^2 + \alpha_n||Sx_{n+1} - Sy_n||,
\]
and it now follows as in the proof of Theorem 1 that \( \lim_{n \to \infty} ||x_{n+1} - x^*|| = 0 \), completing the proof of Theorem 2.

**COROLLARY 2**

Let \( E \) and \( T \) be as in Theorem 2. Let \( \{\alpha_n\} \) be as in Corollary 1. Then the sequence \( \{x_n\} \) generated from an arbitrary \( x_0 \in E \) by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + (I - T)x_n), \quad n \geq 0
\]
converges strongly to the solution of the equation \( Tx = f \).

**Remark 4** Theorem 2 extends and generalizes Theorem 2 of of Chidume [9] to arbitrary real Banach spaces and to uniformly continuous strongly accretive operators.

**Remark 5** It is interesting to observe that under the hypothesis of Theorem 2 the range of \( T \) is bounded if and only if \( E \) is bounded whereas the range of \( (I - T) \) may be bounded even if \( E \) is not. A simple example of an operator satisfying the hypothesis of Theorem 2 and having an unbounded range is the following:

**EXAMPLE** Let \( \mathbb{R} \) denote the reals with the usual norm. Define \( T : \mathbb{R} \to \mathbb{R} \) by
\[
Tx = x - \frac{1}{2}\cos x.
\]

Then \( T \) is Lipschitz and strongly accretive. Furthermore, the range of \( (I - T) \) is bounded. Note that the range of \( T \) is unbounded.

As a consequence of Remark 5, Theorem 2 is an improvement of Theorems 5 and 6 of Chidume [10] in which the range of \( T \) is assumed to be bounded.
2.3. Convergence Theorems for the Equation $x + Tx = f$

**THEOREM 3**

Suppose $E$ is an arbitrary real Banach space and $T : E \rightarrow E$ is a uniformly continuous accretive operator. Suppose the range of $T$ is bounded and $\{\alpha_n\}$ and $\{\beta_n\}$ are as in Theorem 1. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in E$ by

$$
y_n = (1 - \beta_n)x_n + \beta_n(f - Tx_n), \quad n \geq 0
$$

$$
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f - Ty_n), \quad n \geq 0
$$

converges strongly to the solution of the equation $x + Tx = f$.

**PROOF** The existence of a solution to the equation follows from Martin [20] and the uniqueness follows from the accretive condition of $T$. Define $S : E \rightarrow E$ by $Sx = f - Tx$. Let $x^*$ denote the solution of the equation $x + Tx = f$. Then $x^*$ is a fixed point of $S$ and $S$ is uniformly continuous. Furthermore,

$$
((I - S)x - (I - S)y, j(x - y)) \geq \|x - y\|^2 \forall x, y \in E.
$$

Thus as in the proof of Theorem 1 we obtain

$$
\|x_{n+1} - x^*\| \leq [1 - \alpha_n + \alpha_n^2]\|x_n - x^*\| + \alpha_n^2\|x_n - Sy_n\| + \alpha_n\|Sx_{n+1} - Sy_n\|.
$$

Let

$$
D := \sup\{|Sy_n - x^*| + \|Sx_n - x^*\|, n \geq 0\} + \|x_0 - x^*\|.
$$

Then as in the proof of Theorem 2 we obtain $\|x_n - x^*\| \leq D, \ n \geq 0$,

$$
\|y_n - x^*\| \leq D, \ n \geq 0, \ |x_n - Sy_n| \leq 2D, \ n \geq 0 \text{ and } \lim_{n \to \infty} \|Sx_{n+1} - Sy_n\| = 0.
$$

Thus we obtain

$$
\|x_{n+1} - x^*\| \leq [1 - \alpha_n + \alpha_n^2]\|x_n - x^*\| + 2\alpha_n^2D + \alpha_n\|Sx_{n+1} - Sy_n\|
$$

$$
\leq [1 - \alpha_n]\|x_n - x^*\| + 3\alpha_n^2D + \alpha_n\|Sx_{n+1} - Sy_n\|
$$

and it follows as in the proof of Theorem 1 that $\lim_{n \to \infty} \|x_{n+1} - x^*\| = 0$, completing the proof of Theorem 3.
COROLLARY 3

Suppose $E$ and $T$ are as in Theorem 3. Let $\{\alpha_n\}$ be as in Corollary 1. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in E$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f - T x_n), \quad n \geq 0$$

converges strongly to the solution of the equation $x + Tx = f$.

We conclude this paper with the following example of an $m$-accretive operator with a bounded range.

Example ([1], Corollary 3.2, [29], p. 415) Let $\mathbb{R}$ denote the reals with the usual norm. Define $T : E \to E$ by

$$Tx = \begin{cases} 
-1, & x \in (-\infty, -1) \\
-\sqrt{1 - (x+1)^2}, & x \in [-1, 0) \\
\sqrt{1 - (x-1)^2}, & x \in [0, 1] \\
1, & x \in (1, \infty)
\end{cases}$$

It has been shown in [1] that $T$ has the desired property.

Remark 6 It is clear that the hypothesis that the ranges of $(I - T)$ and $T$ are bounded imposed in Theorems 2 and 3 respectively can be replaced by the assumption that the sequences $\{Sy_n\}$ and $\{Sx_n\}$ are bounded.

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References


