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Ishikawa Iteration Process for Nonlinear Lipschitz Strongly Accretive Mappings

C. E. CHIDUME*

International Centre for Theoretical Physics, Trieste, Italy

AND

M. O. OSILIKE

Department of Mathematics, University of Nigeria, Nsukka, Nigeria

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Let $E = L_p$, $p \geq 2$ and let $T: E \to E$ be a Lipschitzian and strongly accretive mapping. Let $S: E \to E$ be defined by $Sx = f - Tx + x$. It is proved that under suitable conditions on the real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$, the iteration process, $x_0 \in E$, $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S[(1 - \beta_n)x_n + \beta_n Sx_n]$, $n \geq 0$, converges strongly to the unique solution of $Tx = f$. A related result deals with the iterative approximation of fixed points for Lipschitz strongly pseudocontractive mappings in E. A consequence of our result gives an affirmative answer to a problem posed by C. E. Chidume (*J. Math. Anal. Appl.* **151,** No. 2 (1990), 453–461). © 1995 Academic Press, Inc.

1. INTRODUCTION

Let E be a real normed linear space. A mapping T with range $R(T)$ in E is called *accretive* [3] if for each x, y in E and all real numbers $t > 0$ the following inequality is satisfied:

$$\|x - y\| \leq \|x - y + t(Tx - Ty)\|. \tag{1}$$

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If E is a Hilbert space the accretive condition (1) reduces to

$$\langle Tx - Ty, x - y \rangle \geq 0,$$

for all x, y in E. The accretive operators were introduced in 1967 by Browder [3] and Kato [20]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \qquad U(0) = U_0,$$

is solvable if T is locally Lipschitzian and accretive on E. The accretive operators are intimately connected with the important class of *pseudocontractive* mappings. For K a subset of E, a mapping $T: K \to K$ is said to be a *strong pseudocontraction* if there exists $t > 1$ such that the inequality,

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|$$

holds for all x, y in K and $r > 0$. If, in the above definition, $t = 1$, then T is said to be a *pseudocontractive* mapping. Pseudocontractive mappings have been studied by various authors (see e.g., [1, 3, 5, 8, 20]). Interest in such mappings stems mainly from the fact that a mapping T is pseudocontractive if and only if $(I - T)$ is accretive [3, Proposition 1]. The mapping theory for accretive operators is thus closely related to the fixed point theory of pseudocontractive mappings.

In [4], Browder proved the following surjectivity result: If $T: E \to E$ is locally Lipschitz and accretive then T is *m-accretive*; i.e., the map $(I + T)$ is surjective. This result was subsequently generalized by Martin [23] to *continuous* accretive operators. An obvious consequence of this result is that the equation,

$$x + Tx = h, \tag{2}$$

for a given h in E, has a solution. Zarantonello [36] proved that, if H is a Hilbert space, the solution is unique. In fact, it is easy to see that if T is accretive on any Banach space E and Eq. (2) has a solution, then the solution is necessarily unique.

For a Banach space E, let J denote the normalised duality mapping from E to $2E^*$ given by

$$Jx = \{f^* \in E^*: \|f^*\|^2 = \|x\|^2 = \langle x, f^* \rangle\},$$

where E^* denotes the dual space of E and $\langle\ ,\ \rangle$ denotes the generalised duality pairing. It is well known that if E^* is strictly convex, then J is single-valued and if E^* is uniformly convex, then J is uniformly continuous on bounded sets (see, e.g., [4, 33]).

A strong notion of accretiveness has been introduced. A mapping T of E into E is said to be *strongly accretive* (see, e.g., [1]) with constant k if for each x, y in E there exists $\omega \in J(x - y)$ such that

$$\langle Tx - Ty, \omega \rangle \geq k \|x - y\|^2, \tag{3}$$

for some real constant $k > 0$. Without loss of generality we may assume that $k \in (0, 1)$. Strongly accretive mappings have been studied extensively by various authors (see, e.g., [1, 4, 17, 24, 26, 27]). In [1], Bogin considered the connection between strong pseudocontractions and strongly accretive operators. He proved that T is a strong pseudocontraction if and only if $(I - T)$ is strongly accretive. He then proved a fixed point theorem of Browder [4] for Lipschitz strongly accretive operators. Since $(T - kI)$ is accretive if T is strongly accretive with constant k, it follows from Martin [23] (see also Morales [24]) that if E is a Banach space and $T: E \to E$ is continuous and strongly accretive then T is surjective. Consequently the equation

$$Tx = f, \tag{4}$$

for a given f in E has a solution in E.

Methods for approximating solutions of Eqs. (2) and (4) (when they are known to exist) have been investigated by several authors. For Eq. (2), if E is a Hilbert space and T is monotone and Lipschitzian with Lipschitz constant 1, Dotson [14] proved that an iteration process of the type introduced by Mann (now generally called the *Mann iteration process*; see, e.g., [22]) converges strongly to the unique solution of Eq. (2). This result has been extended, in a series of papers, by one of the authors (see [9, 11, 12]) to the case where E is now any real Banach space with a uniformly convex dual space, E^*.

It is an open question (see, e.g., [8, 32]) whether or not the Mann iteration process converges to a fixed point of $T: C \to C$ if T is a Lipschitz pseudocontractive mapping of C into itself, where C is a subset of a Hilbert space and T has a fixed point in C. To iteratively approximate fixed points of Lipschitz pseudocontractive mapping, Ishikawa [18] introduced an iteration process (now generally referred to as the *Ishikawa iteration process*) which, in a sense, is more general than the Mann iteration process and which, under suitable conditions, converges to a fixed point of Lipschitz pseudocontractive mappings in Hilbert spaces (when a fixed point is known to exist). The Mann iteration process and the Ishikawa iteration process are now employed in approximating solutions of several nonlinear operators equations in Banach spaces (see, e.g., [5, 7, 8–17, 19, 22, 24–33]).

More recently, the Mann iteration method has been applied to nonexpansive and pseudocontractive maps in hyperbolic spaces and excellent results have been obtained (see, e.g., [2, 31]).

In [32] Rhoades compared the performance of these two iteration processes. He showed that even though the iteration processes are similar, they may exhibit different behaviors for different classes of nonlinear mappings. Thus, it is of interest to examine the behaviour of the two processes for any given class of nonlinear mappings.

For the operator equation (4), one of the authors [12] has proved that the Mann iteration process converges to a solution of the equation when $E = L_p, p \geq 2$. As a consequence of this he proved [12, Theorem 2] that if C is a closed convex and bounded subset of E and $T: C \to C$ is a *Lipschitz strongly pseudocontractive* mapping of C into itself then the Mann iteration process converges strongly to the unique fixed point of T. He then posed the natural question [12, Question 2, p. 460], "Can the Ishikawa iteration process be employed for approximating a solution of Eq. (4) if T is Lipschitz and strongly accretive and $E = L_p, p \geq 2$?"

It is our purpose in this paper to give an affirmative answer to this question by proving that if $E = L_p, p \geq 2$, the Ishikawa iteration process converges strongly to a solution of Eq. (4) when $T: E \to E$ is Lipschitz and strongly accretive. As an immediate consequence of this result we prove that the Ishikawa process can be used to approximate the fixed point of a Lipschitz strongly pseudocontractive maps in E, when such a fixed point is known to exist, thus resolving in the affirmative Question 2 of [12, p. 460].

2. Preliminaries

In the sequel we shall make use of the following results:

A Banach space E is called an *upper weak parallelogram space with constant* $b > 0$ if

$$\|x + y\|^2 + b\|x - y\|^2 \geq 2\|x\|^2 + 2\|y\|^2 \tag{5}$$

holds for all x, y in E. If L_p has at least two disjoint sets of positive finite measure, it is proved in [6] that L_p spaces, $p \geq 2$ are upper weak parallelogram spaces with $(p - 1)$ as the smallest number b such that inequality (5) is satisfied for all x, y in $L_p, p \geq 2$.

In the sequel we shall make use of a characterization of upper weak parallelogram spaces in terms of normalized duality mapping given in the following results.

Theorem B (Bynum [6]). *Let E be a Banach space with normalised duality mapping, J. Then E is an upper weak parallelogram space with* *constant* $b > 0$ *if and only if, for each* x, y *in* E, $j \in Jy$,

$$\|x + y\|^2 \leq b\,\|x\|^2 + \|y\|^2 + 2\langle x, j\rangle. \tag{6}$$

For $E = L_p$, $p \geq 2$, J is single-valued and inequality (6) can be re-stated as (see, e.g., [9, 12])

$$\|x + y\|^2 \leq (p - 1)\,\|x\|^2 + \|y\|^2 + 2\langle x, j(y)\rangle. \tag{7}$$

For the rest of this paper, $E = L_p$, $p \geq 2$, where L_p is assumed to have at least two disjoint sets of positive finite measure, and the single-valued duality map is denoted by j. The Lipschitz constant of T is denoted by $L(\geq 1)$ and the constant appearing in the definition of a strongly accretive map is denoted by $k \in (0, 1)$.

3. Main Results

We prove the following theorems.

THEOREM 1. *Let* $T: E \to E$ *be a Lipschitzian and strongly accretive map, and let* $S: E \to E$ *be defined by* $Sx = f - Tx + x$. *Let* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *be real sequences satisfying:*

(i) $0 \leq \beta_n \leq \alpha_n \leq k(\omega + 2k - 1)^{-1}$ *for each* n, *where* $\omega = (p - 1)L_*^4 + 2L_*(1 + L_*)$ *and* $L_* = 1 + L$.

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, given $f \in E$, for arbitrary $x_0 \in E$, the iteration process,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S[(1 - \beta_n)x_n + \beta_n Sx_n], \qquad n \geq 0, \tag{8}$$

converges strongly to the unique solution of $Tx = f$.

Moreover, if $\alpha_n = k[\omega + 2k - 1]^{-1}$ for all n, then

$$\|x_{n+1} - x^*\| \leq \rho^{n/2}\,\|x_1 - x\|,$$

where x^* denotes the solution of $Tx = f$ and

$$\rho = 1 - k^2[\omega + 2k - 1]^{-1} \in (0, 1).$$

Proof. The existence of a solution to $Tx = f$ follows from Morales [24]. Let x^* be a solution. Observe that x^* is a fixed point of S and that S is Lipschitz with constant $L_* = (1 + L)$. Moreover, for each x, y in E,

$$\langle Sx - Sy, j(x - y)\rangle \leq (1 - k)\|x - y\|^2. \tag{9}$$

Rewrite (8) as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n, \qquad n \geq 0, \tag{10}$$

where

$$y_n = (1 - \beta_n)x_n + \beta_n S x_n, \qquad n \geq 0. \tag{11}$$

Using (7), (9), (10), and (11) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*)\|^2 \\ &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + (p - 1)\alpha_n^2 L_*^2 \|y_n - x^*\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n)\langle Sy_n - x^*, j(x_n - x^*)\rangle \end{aligned}$$

and

$$\begin{aligned} \|y_n - x^*\| &= \|(1 - \beta_n)(x_n - x^*) + \beta_n(Sx_n - x^*)\| \\ &\leq [1 + \beta_n(L_* - 1)] \|x_n - x^*\| \\ &\leq L_* \|x_n - x^*\|, \qquad \text{since } \beta_n < 1. \end{aligned} \tag{12}$$

Also,

$$\langle Sy_n - x^*, j(x_n - x^*)\rangle = \langle Sy_n - Sx_n, j(x_n - x^*)\rangle + \langle Sx_n - x^*, j(x_n - x^*)\rangle$$

and

$$\begin{aligned} \langle Sy_n - Sx_n, j(x_n - x^*)\rangle &\leq \|Sy_n - Sx_n\| \, \|x_n - x^*\| \\ &\leq L_* \|\beta_n(Sx_n - x_n)\| \, \|x_n - x^*\| \\ &\leq L_*\beta_n[\|Sx_n - x^*\| + \|x_n - x^*\|] \, \|x_n - x^*\| \\ &\leq L_*\beta_n(1 + L_*) \|x_n - x^*\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} \langle Sy_n - x^*, j(x_n - x^*)\rangle &\leq L_*\beta_n(1 + L_*) \|x_n - x^*\|^2 + \langle Sx_n - x^*, j(x_n - x^*)\rangle \\ &\leq L_*\beta_n(1 + L_*) \|x_n - x^*\|^2 + (1 - k) \|x_n - x^*\|^2 \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)^2 \|x_n - x^*\|^2 + (p - 1)\alpha_n^2 L_*^4 \|x_n - x^*\|^2 \\ &\quad + 2\alpha_n(1 - \alpha_n)[L_*\beta_n(1 + L_*) \|x_n - x^*\|^2 \end{aligned}$$

$$+ (1-k)\|x_n - x^*\|^2] \leq [(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)(1-k)$$
$$+ (p-1)\alpha_n^2 L_*^4 + 2\alpha_n^2 L_*(1+L_*)]\|x_n - x^*\|^2,$$

since $\beta_n \leq \alpha_n$, $(1-\alpha_n) < 1$,

$$= [(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)(1-k) + \alpha_n^2\omega]\|x_n - x^*\|^2,$$
$$= [1 - 2\alpha_n k + \alpha_n^2(\omega + 2k - 1)]\|x_n - x^*\|^2$$
$$\leq [1 - 2\alpha_n k + \alpha_n k]\|x_n - x^*\|^2, \qquad \text{by condition (i)}$$
$$= [1 - \alpha_n k]\|x_n - x^*\|^2,$$

so that

$$\|x_{n+1} - x^*\|^2 \leq \exp(-k\alpha_n)\|x_n - x^*\|^2. \tag{13}$$

Iterating the last inequality from $n = 1$ to N, we obtain

$$\|x_{N+1} - x^*\|^2 \leq \exp\left(-k\sum_{n=1}^{N}\alpha_n\right)\|x_1 - x^*\|^2 \to 0 \qquad \text{as } N \to \infty,$$

since $\sum\alpha_n = \infty$. Hence $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^*.

Error Estimate. If $\alpha_n = k(\omega + 2k - 1)^{-1}$ for each n, then

$$\|x_{n+1} - x^*\|^2 \leq (1 - k^2[\omega + 2k - 1]^{-1})\|x_n - x^*\|^2$$
$$\leq (1 - k^2[\omega + 2k - 1]^{-1})^n\|x_1 - x^*\|^2$$

so that

$$\|x_{n+1} - x^*\| \leq \rho^{n/2}\|x_1 - x^*\|,$$

where

$$\rho = 1 - k^2[\omega + 2k - 1]^{-1} \in (0, 1).$$

COROLLARY 1. *Let E, T, and S be as in Theorem 1, and let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying:*

(i) $0 \leq \alpha_n \leq k[\omega + 2k - 1]^{-1}$ *for each n,*
(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Then given $f \in E$, for arbitrary $x_0 \in E$, the iteration process

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S x_n, \qquad n \geq 0,$$

converges strongly to the unique solution of $Tx = f$.

THEOREM 2. *Let C be a nonempty closed convex and bounded subset of E, and $T: C \to C$ a Lipschitz strongly pseudocontractive mapping with constant t. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying:*

(i) $0 \le \beta_n \le \alpha_n \le k^*[\omega + 2k^* - 1]^{-1}$, *where* $k^* = (t-1)/t \in (0, 1)$ *and* $\omega = (p-1)L^4 + 2L(1+L) > 1$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, for arbitrary $x_0 \in E$, the iteration process,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n], \qquad n \ge 0,$$

converges strongly to the unique fixed point of T.

Proof. The existence of a fixed point follows from Deimling [13]. Let x^* denote a fixed point of T. Since T is strongly pseudocontractive we have (see, e.g., [1, 8])

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \frac{t-1}{t}\|x-y\|^2 \tag{14}$$

for each x, y in C.

Set $k^* = (t-1)/t$ and $y_n = (1-\beta_n)x_n + \beta_n T x_n$. Using (7) we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\|^2 \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + (p-1)\alpha_n^2 L^2\|y_n - x^*\| \\ &\quad + 2\alpha_n(1-\alpha_n)\langle Ty_n - x^*, j(x_n - x^*)\rangle\end{aligned}$$

and

$$\begin{aligned}\|y_n - x^*\| &= \|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\| \\ &\le [1 + \beta_n(L-1)]\,\|x_n - x^*\| \\ &\le L\|x_n - x^*\|.\end{aligned}$$

Moreover,

$$\langle Ty_n - x^*, j(x_n - x^*)\rangle = \langle Ty_n - Tx_n, j(x_n - x^*)\rangle + \langle Tx_n - x^*, j(x_n - x^*)\rangle$$

and

$$\begin{aligned}\langle Ty_n - Tx_n, j(x_n - x^*)\rangle &\le \|Ty_n - Tx_n\|\,\|x_n - x^*\| \\ &\le L\|\beta_n(Tx_n - x_n)\|\,\|x_n - x^*\| \\ &\le L\beta_n(1+L)\|x_n - x^*\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\langle Ty_n - x^*, j(x_n - x^*)\rangle &\leq L\beta_n(1 + L)\|x_n - x^*\|^2 + \langle Tx_n - x^*, j(x_n - x^*)\rangle \\ &= L\beta_n(1 + L)\|x_n - x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad - \langle (I - T)x_n - (I - T)x^*, j(x_n - x^*)\rangle \\ &\leq L\beta_n(1 + L)\|x_n - x^*\|^2 + (1 - k^*)\|x_n - x^*\|^2.\end{aligned}$$

Finally,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq [(1 - \alpha_n)^2 + 2\alpha_n(1 - \alpha_n)(1 - k^*) + (p - 1)\alpha_n^2 L^4 \\ &\quad + 2\alpha_n^2 L(1 + L)]\|x_n - x^*\|^2 \\ &= [1 - 2\alpha_n k^* + \alpha_n^2(\omega + 2k^* - 1)]\|x_n - x^*\|^2, \\ &\leq [1 - 2\alpha_n k^* + \alpha_n k^*]\|x_n - x^*\|^2, \qquad \text{by condition (i)} \\ &= [1 - \alpha_n k^*]\|x_n - x^*\|^2,\end{aligned}$$

and the result follows as in the proof of Theorem 1.

COROLLARY 2. *Let E, C, and T be as in Theorem 2, and let* $\{\alpha_n\}_{n=0}^{\infty}$ *be a real sequence satisfying*

(i) $0 \leq \alpha_n \leq k^*[\omega + 2k^* - 1]^{-1}$, $n \geq 0$, *where* $k^* = (t - 1)/t$, *and* $\omega = (p - 1)L^4 + 2L(1 + L)$.

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, for arbitrary $x_0 \in E$, *the iteration process,*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \qquad n \geq 0,$$

converges strongly to the unique fixed point of T.

Remark 1. Theorems 1 and 2 resolve in the affirmative Problem 2 of Chidume [12, p. 460].

Remark 2. The results of this paper and those of [12] show that either the Mann or the Ishikawa iteration process can be used to approximate solutions of nonlinear equations for Lipschitz strongly accretive and strongly pseudocontractive mappings in L_p spaces, $p \geq 2$. Furthermore, the error estimates obtained for the two methods for these classes of nonlinear mappings are of the same order, so that for computational purposes, the Mann process may have some advantage due to its simplicity.

THEOREM TX1 [35, Theorem 3.1, p. 13]. *Let* $1 < q \leq 2$ *and E be a q-uniformly smooth Banach space. Let* $T: E \to E$ *be a Lipschitz strongly* *accretive operator. Define* $S: E \to E$ *by* $Sx = f - Tx + x$. *Suppose that* $\{c_n\}_{n=0}^{\infty}$ *is a sequence of reals satisfying:*

(i) $0 < c_n \leq t_q$ *for all* $n \geq 1$, *where* t_q *is the (smaller) solution of the equation*

$$f(t) := q(q-1)(1-k)t - (1 + d_q L^q)t^{q-1} + \tfrac{1}{2}qk = 0, \qquad t > 0, \quad (16)$$

with L *the Lipschitz constant of* S *and* d_q *the constant appearing in inequality* (15).

(ii) $\sum_n c_n = \infty$.

Then for arbitrary $x_0 \in E$, *the Mann sequence,*

$$x_{n+1} = (1 - c_n)x_n + c_n S x_n, \qquad n \geq 0,$$

converges strongly to the unique solution x^* *of the equation* $Tx = f$. *Moreover, if* $c_n = t_q$ *for all* $n \geq 1$, *then*

$$\|x_{n+1} - x^*\| \leq \rho^{n/q} \|x_1 - x^*\|,$$

where $\rho = 1 - \frac{1}{2} qkt_q \in (0, 1)$.

THEOREM Tx2 [35, Theorem 3.2, p. 16]. *Let* K *be a nonempty bounded closed convex subset of a q-uniformly smooth Banach space* E, $1 < q \leq 2$, *and* $T: K \to K$ *be a Lipschitz strongly pseudocontractive mapping with a constant* $t > 1$ *and a Lipschitz constant* L. *Let* $\{c_n\}_{n=1}^{\infty}$ *be a sequence of real numbers satisfying the properties* (i) *and* (ii) *of Theorem TX1 with* $k = (t-1)/t$. *Then for a given* $x_0 \in K$, *the Mann sequence*

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n, \qquad n \geq 0,$$

converges strongly to the unique fixed point of T.

Remark 3. Corollaries 3.1 and 3.2 of [35] are special cases respectively of Theorems TX1 and TX2 in which $E = L_p$ $(1 < p < 2)$.

Remark 4. It is clear that condition (i) of Theorem TX1 (which is also imposed in Theorem TX2 and in Corollary 3.1 and Corollary 3.2 of [35]) is not convenient for applications since t_q is connected with a solution of Eq. (16).

In the following theorems we prove convergence theorems which eliminate this problem completely.

THEOREM 3. *Let* E *be a q-uniformly smooth Banach space and* $T: E \to E$ *be a Lipschitz strongly accretive operator. Define* $S: E \to E$

by $Sx = f - Tx + x$. *Suppose* $\{c_n\}_{n=0}^{\infty}$ *is a sequence of real numbers satisfying the following conditions:*

(i) $\lim c_n = 0$

(ii) $\sum_n c_n = \infty$.

Then, for arbitrary $x_0 \in E$, *the Mann sequence* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = (1 - c_n)x_n + c_n S x_n, \qquad n \geq 0,$$

converges strongly to the unique solution of $Tx = f$. *Moreover, if* $c_n = 1/(n+1)k(1-k)$ *for all* $n \geq 1$, *and* x^* *is the unique solution, then*

$$\|x_{n+1} - x^*\| = O(n^{-1/q}).$$

Proof. The existence of a solution follows from Morales [24] and uniqueness follows from the strong accretivity of T. Let x^* denote the solution. Observe that x^* is a fixed point of S. Clearly, for arbitrary x, $y \in E$, we have

$$\langle Sx - Sy, J_q(x - y)\rangle \leq (1 - k)\,\|x - y\|^q.$$

If L denotes the Lipschitz constant of S we have

$$\begin{aligned} \|x_{n+1} - x^*\|^q &= \|(1 - c_n)(x_n - x^*) + c_n(Sx_n - x^*)\|^q \\ &\leq [(1 - c_n)^q + q(1 - k)c_n(1 - c_n)^{q-1} + d_q L^q c_n^q]\,\|x_n - x^*\|^q. \end{aligned} \tag{17}$$

Condition (i) implies that there exists an integer $N_0 > 0$ such that for all $n \geq N_0$,

$$d_q L^q c_n^{q-1} \leq k^2. \tag{18}$$

For arbitrary $x > 0$, consider the function

$$h(x) = (1 + x)^q, \qquad q > 1.$$

Then, there exists $\xi \in (0, x)$ such that

$$h(x) = h(0) + xh'(x) + \frac{x^2}{2}h''(\xi).$$

Observe that $h''(\xi) \geq 0$ so that the last equation reduces to

$$h(0) + xh'(x) \leq h(x).$$

Put $x = (1 - k)c_n(1 - c_n)^{-1}$ in this inequality to get

$$(1 - c_n)^q + q(1 - k)c_n(1 - c_n)^{q-1} \leq (1 - kc_n)^q \leq (1 - kc_n), \quad (19)$$

since $q > 1$. Substitution of (18) and (19) in (17) yields

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\leq [1 - k(1 - k)c_n]\,\|x_n - x^*\|^q \qquad \text{for all } n \geq N_0 \\ &\leq \exp\left(-k(1 - k)\sum_{i=N_0}^{N} c_i\right)\|x_1 - x^*\|^q \to 0 \qquad \text{as } N \to \infty,\end{aligned}$$

using Condition (ii). Hence $x_n \to x^*$ as $n \to \infty$. The error estimate follows as in [10]. The proof is complete.

THEOREM 4. *Let K be a nonempty bounded closed convex subset of a q-uniformly smooth Banach space E and $T: K \to K$ be a Lipschitz strongly pseudocontractive mapping with constant $t > 1$ and a Lipschitz constant L. Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of real numbers satisfying the following conditions:*

(ii) $\lim_{n\to\infty} c_n = 0$

(ii) $\sum_n c_n = \infty$.

Then for a given $x_0 \in K$, the Mann sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - c_n)x_n + c_nTx_n, \; n \geq 0,$$

converges strongly to the unique fixed point of T.

Proof. The existence and uniqueness of a fixed point follow from Deimling [13] and the strong pseudocontractivity of T, respectively. Let $k = (t - 1)/t$ and let x^* denote the unique fixed point of T. Clearly,

$$\langle (I - T)x - (I - T)y, \quad J_q(x - y)\rangle \geq k\,\|x - y\|^q$$

for all x, $y \in K$. Proceeding as in the proof of Theorem 3 we obtain the estimates

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\leq (1 - c_n)^q\,\|x_n - x^*\|^q + qc_n(1 - c_n)^{q-1}\langle Tx_n - Tx^*, J_q(x_n - x^*)\rangle \\ &\quad + d_q c_n^q\,\|Tx_n - Tx^*\|^q \\ &= (1 - c_n)^q\,\|x_n - x^*\|^q - qc_n(1 - c_n)^{q-1}\langle x_n - Tx_n, J_q(x_n - x^*)\rangle \\ &\quad + qc_n(1 - c_n)^{q-1}\,\|x_n - x^*\|^q + d_q c_n^q L^q\,\|x_n - x^*\|^q \\ &\leq [(1 - c_n)^q + q(1 - k)c_n(1 - c_n)^{q-1} + d_q c_n^q L^q]\,\|x_n - x^*\|^q,\end{aligned}$$

which is exactly inequality (17). The rest of the argument now follows as in the proof of Theorem 3 to yield that $x_n \to x^*$ as $n \to \infty$. The proof is complete.

Remark 5. It is known (see, e.g., [34, p. 1131]) that

$$L_p \text{ (or } l_p\text{) is } \begin{cases} p - \text{uniformly smooth}, & \text{if } 1 < p \leq 2, \\ 2 - \text{uniformly smooth}, & \text{if } p > 2. \end{cases}$$

It then follows that the following corollaries are immediate consequences of Theorems 3 and 4, respectively.

COROLLARY 3. *Suppose that* $E = L_p$ (or l_p), $1 < p < \infty$, *and* T, S, *and* c_n *are as in Theorem* 3. *Then the Mann sequence* $\{x_n\}_{n+0}^{\infty}$ *defined by* $x_0 \in E$,

$$x_{n+1} = (1 - c_n)x_n + c_n S x_n, \qquad n \geq 0,$$

converges strongly to the unique solution of the equation $Tx = f$.

COROLLARY 4. *Let* $E = L_p$ *(or* l_p*)*, $1 < p < \infty$, K *be a bounded closed convex subset of* E, and $T: K \to K$ *a Lipschitz and strongly pseudocontractive mapping with a constant* $t > 1$ *and a Lipschitz constant* L. *Let* c_n *be as in Theorem* 4. *Then for a given* $x_0 \in K$, *the Mann sequence*

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n, \qquad n \geq 0,$$

converges strongly to the unique fixed point of T.

Remark 6. Corollaries 3 and 4 which are valid for all L_p (or l_p) with $1 < p < +\infty$ are improvements of Corollaries 3.1 and 3.2 of [35] which are proved for L_p spaces, where $1 < p < 2$. Moreover, the iteration parameter c_n in our theorems can be chosen at the beginning of the iteration process independent of finding a solution of Eq. (16) as is required in the theorems of [35]. A prototype for our c_n is $c_n = 1/(n + 1)k(1 - k)$ for all n.

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Notes added in proof. Long after we submitted this paper, Tan and Xu [35] independently resolved Problems 1 and 2 of Chidume [12] in the affirmative in the more general setting of

q-uniformly smooth Banach spaces, $q > 1$. In such spaces E the

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q \|y\|^q$$

inequality is known to hold for all $x, y \in E$, where for arbitrary $x \in E$

$$J_q(x) = \{f^* \in E^* : \langle x, f\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

and $J_q(x) = \|x\|^{q-2}J(x)$ for $x \in E$, $x \neq 0$, and where d_q is a constant depending on q (see e.g., [35]). Let $k \in (0, 1)$ denote the constant of strong accretivity of the operator T. Tan and Xu also proved the results which are affirmative answers to Problem 1 of [12].

REFERENCES

1. J. BOGIN, "On Strict Pseudocontractions and a Fixed Point Theorem," Technion Preprint Series No. MT-219, Haifa, Israel, 1974.
2. J. BORWEIN, S. REICH, AND I. SHAFRIR, Krasnoseski–Mann iterations in normed spaces, *Canad. Math. Bull.* **35** (1992), 21–28.
3. F. E. BROWDER, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, *Bull. Amer. Math. Soc.* **73** (1967), 875–882.
4. F. E. BROWDER, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *Proc. Sympos. Pure Math.* **18** (1976).
5. F. E. BROWDER AND W. V. PETRYSHYN, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.* **20** (1967), 197–228.
6. W. L. BYNUM, Weak parallelogram laws for Banach spaces, *Canad. Math. Bull.* **19,** No. 3 (1976), 269–275.
7. R. E. BRUCK, JR., The iterative solution of the equation $f \in x + Tx$ for a monotone operator T in Hilbert space, *Bull. Amer. Math. Soc.* **79** (1973), 1258–1262.
8. C. E. CHIDUME, Iterative approximation of fixed points of Lipschitzian strictly pseudocontractive mappings, *Proc. Amer. Math. Soc.* **99,** No. 2 (1987), 283–288.
9. C. E. CHIDUME, Iterative solution of nonlinear equations of the monotone and dissipative types, *Appl. Anal.* **33** (1989), 79–86.
10. C. E. CHIDUME, An approximation method for monotone Lipschitzian operators in Hilbert spaces, *J. Austral. Math. Soc. Ser. A* **41** (1986), 59–63.
11. C. E. CHIDUME, Iterative solution of nonlinear equations of the monotone type in Banach spaces, *Bull. Austral. Math. Soc.* **42** (1990), 21–31.
12. C. E. CHIDUME, An iterative process for nonlinear Lipschitzian strongly accretive mappings in L_p spaces, *J. Math. Anal. Appl.* **151,** No. 2 (1990), 453–461.
13. K. DEIMLING, Zeros of accretive operators, *Manuscripta Math.* **13** (1974), 283–288.
14. W. G. DOTSON, An iterative process for nonlinear monotonic nonexpansive operators in Hilbert space, *Math. Comp.* **32,** No. 151 (1978), 223–225.
15. J. C. DUNN, Iterative construction of fixed points for multivalued operators of the monotone type, *J. Funct. Anal.* **27** (1978), 38–50.
16. M. EDELSTEIN AND R. C. O'BRIAN, Nonexpansive mappings, asymptotic regularity and successive approximations, *J. London Math. Soc.* (2) **17,** No. 3 (1978), 547–554.
17. J. GWINNER, On the convergence of some iteration processes in uniformly convex Banach spaces, *Proc. Amer. Math. Soc.* **81** (1978), 29–35.
18. S. ISHIKAWA, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.* **149** (1944), 147–150.

19. S. ISHIKAWA, Fixed points and iteration of a nonexpansive mappings in a Banach space, *Proc. Amer. Math. Soc.* **73** (1976), 65–71.
20. T. KATO, Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan* **18** (1967), 508–520.
21. W. A. KIRK AND C. MORALES, On the approximation of fixed points of locally nonexpansive mappings, *Canad. Math. Bull.* **24,** No. 4 (1981), 441–445.
22. W. R. MANN, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953), 506–510.
23. R. H. MARTIN, JR., A global existence theorem for autonomous differential equations in Banach spaces, *Proc. Amer. Math. Soc.* **26** (1970), 307–314.
24. C. MORALES, Surjectivity theorems for multi-valued mappings of accretive type, *Comment. Math. Univ. Carolin.* **26** (1985), 2.
25. R. N. MUKERJEE, Construction of fixed points of strictly pseudocontractive mappings in generalised Hilbert spaces and related application, *Indian J. Pure Appl. Math.* **15** (1966), 276–284.
26. O. NEVALINNA AND S. REICH, Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces, *Israel J. Math.* **32** (1979), 44–58.
27. W. V. PETRYSHYN, Construction of fixed points of demi-compact mappings in Hilbert space, *J. Math. Anal. Appl.* **14** (1966), 276–284.
28. S. REICH, Constructing zeros of accretive operators, II, *Appl. Anal.* **9** (1979), 159–163.
29. S. REICH, Constructive techniques for accretive and monotone operators, *in* "Applied Nonlinear Analysis" (V. Lakshmikantham, Ed.), pp. 335–345, Academic Press, New York, 1979.
30. S. REICH, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.* **85** (1980), 287–292.
31. S. REICH AND I. SHAFRIR, Nonexpansive iterations in hyperbolic spaces, *Nonlinear Anal.* **15** (1990), 537–558.
32. B. E. RHOADES, Comments on two fixed point iteration methods, *J. Math. Anal. Appl.* **56** (1976), 741–750.
33. V. L. SMUL'YAN, On the derivation of the norm in a Banach space, *Dokl. Akad. Nauk. SSSR* **27** (1940), 255–258.
34. H.-K. XU, Inequalities in Banach spaces with applications, *Nonlinear Anal.* **16,** No. 12 (1991), 1127–1138.
35. K. K. TAN AND H. K. XU, Iterative solutions to nonlinear equations of strongly accretive operators in Banach spaces, *J. Math. Anal. Appl.* **178** (1993), 9–21.
36. E. H. ZARANTONELLO, "Solving Functional Equations by Contractive Averaging," Technical Report No. 160, U.S. Army Math. Res. Centre, Madison, WI, 1960.