# Approximation Methods for Nonlinear Operator-Equations of the m-Accretive Type

**ARTICLE** *in* JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS · JANUARY 1995

Impact Factor: 1.12 · DOI: 10.1006/jmaa.1995.1014



Micah Osilike
University of Nigeria

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **189**, 225–239 (1995)

# Approximation Methods for Nonlinear Operator Equations of the $m$-Accretive Type

C. E. CHIDUME

*International Centre for Theoretical Physics, Trieste, 34100, Italy*

AND

M. O. OSILIKE

*Department of Mathematics, University of Nigeria, Nsukka, Nigeria*

*Submitted by John Horváth*

Received October 25, 1993

Let $E$ be a real Banach space which is both uniformly convex and uniformly smooth. Let $T: D(T) \subset E \to E$ be $m$-accretive, where the domain of $T$, $D(T)$, is a *proper subset* of $E$. For any $f \in E$, approximation methods are constructed which converge strongly to a solution of the operator equation $x + Tx = f$. Explicit error estimates are obtained. A related result deals with operator equations of the dissipative type. © 1995 Academic Press, Inc.

## 1. INTRODUCTION

Let $E$ be a real Banach space. A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is called *accretive* [2] if the inequality

$$\|x - y\| \leq \|x - y + t(Tx - Ty)\| \tag{1}$$

holds for each $x, y \in D(T)$ and for all $t > 0$. The operator $T$ is said to be *m-accretive* if $T$ is accretive and $(I + rT)(D(T)) = E$ for all $r > 0$, where $I$ denotes the identity operator on $E$. $T$ is called *dissipative* (respectively *m-dissipative*) if $(-T)$ is accretive (respectively $m$-accretive). If $E = H$, a Hilbert space, the accretive condition (1) is equivalent to the *monotonicity* condition for $T$ in the sense of Browder [4] and Minty [19]. Accretive



0022-247X/95 $6.00

operators were introduced independently in 1967 by Browder [2] and Kato [16]. Interest is such mappings stems mainly from their firm connection with the existence theory for nonlinear equations of evolution (see, e.g., [2]). An early fundamental result in the theory of accretive operators, due to Browder [2], states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \qquad u(0) = u_0 \tag{2}$$

is solvable when $T$ is locally Lipschitzian and accretive on $E$. Utilizing the existence result for (2), Browder [2] further proved that if $T$ is locally Lipschitzian and accretive, then $T$ is $m$-accretive. In particular, for any given $f \in E$, the equation

$$x + Tx = f \tag{3}$$

has a solution. In [18], Martin extended these results of Browder by proving that (2) is solvable if $T$ is *continuous* and accretive, and utilizing this result, he proved that if $T$ is continuous and accretive, then $T$ is $m$-accretive.

It is well known (see, e.g., [34]) that many physically significant problems can be modelled in the form of Eq. (2), where $T$ is accretive. Typical examples of how such evolution equations arise are found in models involving either the heat, or wave on the Schrödinger equation.

Methods of approximating a solution of Eq. (3) (when it is known to exist) have been investigated by various authors (e.g., [6–10, 13, 34]). We introduce two iterative schemes which have been widely used for such approximations.

(a) *The Ishikawa Iteration Process* [15, 27] defined as follows: For $K$ a convex subset of a Banach space $E$ and $T$ a mapping of $K$ into itself, the sequence $\{x_n\}_{n=1}^{\infty}$ in $K$ is defined by

$$x_0 \in K \tag{4}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \tag{5}$$

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \qquad n \geq 0, \tag{6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ satisfy $0 \leq \alpha_n \leq \beta_n < 1$ for all $n$,

$$\lim_{n\to\infty} \beta_n = 0 \qquad \text{and} \qquad \sum_{n=0}^{\infty} \alpha_n \beta_n = \infty.$$

(b) *The Mann Iteration Process* [17, 27] which is similar to the Ishikawa iteration process but with $\beta_n \equiv 0$ and different conditions placed on $\{\alpha_n\}_{n=0}^{\infty}$. More precisely, with $E$, $K$, and $x_0$ as in part (a), the Mann iteration process is defined by

$$x_0 \in K \tag{7}$$

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n, \qquad n \geq 0, \tag{8}$$

where $\{c_n\}_{n=0}^{\infty}$ is a real sequence satisfying $c_0 = 1$; $0 \leq c_n < 1$ for all $n \geq 1$ and $\sum_n c_n = \infty$. In some applications, the condition $\sum_n c_n = \infty$ is replaced by $\sum_n c_n(1 - c_n) = \infty$.

The iteration processes described in (a) and (b) have been studied extensively by various authors and have been successfully employed to approximate fixed points of several nonlinear mappings (when these mappings are already known to have fixed points) and to approximate solutions of several nonlinear operator equations in Banach spaces (e.g., [6–10, 13–15, 17, 22, 27]). We remark here that even though the iteration schemes (a) and (b) are similar, they may exhibit different behaviours for different classes of nonlinear mappings (see, e.g., [27]).

In several applications, the operator $T$ of Eq. (3) is, in general, not defined on the whole of $E$. The domain of $T$, $D(T)$, is generally a *proper* subset of $E$. In such a situation, the iteration processes (a) and (b) may not even be well defined. In the case that $E = H$, *a Hilbert space*, this problem has been overcome by introducing the *proximity map*, $P_K : H \to K$, into the iteration processes (see, e.g., [5, 8]), where $K$ is a closed convex subset of $H$ and $P_K$ is the map which sends each $x \in H$ to its nearest point in $K$.

It is well known that in $H$, the map $P_K$ is *nonexpansive* (i.e., $\|P_K x - P_K y\| \leq \|x - y\|$ for each $x, y \in H$) and this fact is central in using the proximity map. Unfortunately, the fact that $P_K$ is nonexpansive in Hilbert spaces also characterizes Hilbert spaces so that this fact is not available in general Banach spaces.

It is our purpose in this paper to study methods of approximating a solution of Eq. (3) in certain Banach spaces $E$ (much more general than Hilbert spaces) under the natural setting that the domain of $T$, $D(T)$ is a proper subset of $E$ and $T$ maps $D(T)$ into $E$. Our approximation methods which are suitable modifications of an iteration method first introduced by Chidume in [8], will have some resemblance with the Ishikawa and Mann iteration processes and will reduce to them when the domain of $T$ is assumed to be the whole of $E$ or when $T$ is a self mapping of a nonempty convex subset of $E$. Thus, our results will extend several known results to include mappings defined on proper subsets of $E$ and having values in $E$, and will also extend several results from Hilbert spaces to the more general Banach spaces to be considered in the sequel.

## 2. Preliminaries

In the sequel we shall need the following preliminaries and results. Let $E$ be a Banach space. We shall denote by $J$ the normalized duality mapping from $E$ to $2^{E^*}$ given by

$$Jx = \{f^* \in E^* : \|f^*\|^2 = \|x\|^2 = \langle x, f^* \rangle\},$$

where $\langle\ ,\ \rangle$ denotes the generalized duality pairing. If $E^*$ is uniformly convex, then $J$ is single-valued and is uniformly continuous on bounded sets. We shall denote the single-valued duality mapping by $j$.

As a consequence of a result of Kato [16], a mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is accretive (see, e.g., [16]) if for each $x$, $y$ in $D(T)$, there exists $j \in J(x - y)$ such that

$$\langle Tx - Ty, j \rangle \geq 0. \tag{9}$$

For $p > 1$, following [30], we shall associate the generalized duality map $J_p$ from $E$ to $2^{E^*}$ defined by

$$J_p(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^p, \|f^*\| = \|x\|^{p-1}\}.$$

Observe that $J_2$ is the usual normalized duality map, $J$ on $E$. It is well known (e.g., [30]) that

$$J_p(x) = \|x\|^{p-2} J(x), \qquad x \neq 0. \tag{10}$$

A Banach space $E$ is called *smooth* if, for every $x \in E$ with $\|x\| = 1$, there exists a unique $f^* \in E^*$ such that $\|f^*\| = f^*(x) = 1$ (see, e.g., [11, p. 21].

The *modulus of smoothness* of $E$ is the function

$$\rho_E : [0, \infty) \to [0, \infty),$$

defined by

$$\rho_E(\tau) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau \right\}.$$

The Banach space $E$ is called *uniformly smooth* (e.g., [31]) if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0,$$

and for $q > 1$, $E$ is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho_E(\tau) \leq c\tau^q, \qquad \tau \in [0, \infty). \tag{11}$$

It is well known (see, e.g., [30]) that

$$L_p(\text{or } l_p) \text{ is } \begin{cases} p - \text{uniformly smooth}, & \text{if } 1 < p \leq 2 \\ 2 - \text{uniformly smooth}, & \text{if } p \geq 2. \end{cases}$$

The Banach space $E$ is called *uniformly convex* if given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$ we have

$$\|\tfrac{1}{2}(x + y)\| \leq 1 - \delta.$$

It is well known that the $L_p$ spaces $(1 < p < \infty)$ are uniformly convex.

In [30], the following result which will be needed in the sequel is proved.

LEMMA 1 [30]. *Let $q > 1$ be a real number and $E$ be a smooth Banach space. Then the following are equivalent.*

(i) *$E$ is q-uniformly smooth,*

(ii) *There is a constant $c > 0$ such that for every $x, y \in E$, the following inequality holds:*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q. \tag{12}$$

The following result will also be needed in what follows:

THEOREM R1 [26]. *Let $E$ be a Banach space which is both uniformly convex and uniformly smooth. Let $T : D(T) \subseteq E \to E$ be m-accretive and let $J_r = (I + rT)^{-1}$. Then, for each $x \in E$ the strong limit $\lim_{r\to 0} J_r(x)$ exists. Denote this strong limit by $Qx$. Then, $Q : E \to \mathrm{cl}(D(T))$ is a nonexpansive retraction of $E$ onto $\mathrm{cl}(D(T))$. (Here $\mathrm{cl}(D(T))$ denotes the closure of the domain of $T$.)*

It is well known (see, e.g., [1]) that under the hypothesis of Theorem 1, $\mathrm{cl}(D(T))$ is convex.

Another result which will be useful in our work is the following one. In [23, p. 89], Reich proved that if $E^*$ is uniformly convex, then there exists a continuous nondecreasing function $b : [0, \infty) \to [0, \infty)$ such that $b(0) = 0$, $b(ct) \leq cb(t)$ for all $c \geq 1$, and for all $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|). \tag{13}$$

*Remark* 1. Nevanlinna and Reich [21] have shown that for any given continuous nondecreasing function $b$ with $b(0) = 0$ sequences $\{\lambda_n\}_{n=0}^{\infty}$ always exist such that (i) $0 < \lambda_n < 1$, $n \geq 0$; (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$; and (iii) $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$. If $E = L_p$ $(1 < p < \infty)$, we can choose any sequence $\{\lambda_n\}_{n=0}^{\infty}$ in $l^s \backslash l^1$, with $s = p$ if $1 < p \leq 2$ and $s = 2$ if $p \geq 2$.

Finally we shall need the following result.

THEOREM 2 (Dunn [14, p. 41]). *Let $\rho_n$ be recursively generated by*

$$\rho_{n+1} = (1 - \mu_n)\rho_n + \sigma_n^2$$

*with $n \geq 1$, $\rho_1 \geq 0$, $\mu_n \in [0, 1]$, $\sum_{n=1}^{\infty} \mu_n = \infty$, and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Then $\rho_n \geq 0$ for all $n \geq 1$, and $\rho_n \to 0$ as $n \to \infty$.*

## 3. MAIN RESULTS

For the rest of the paper, the Lipschitz constant of the operator $T$ will be denoted by $L$; $c > 0$ is the constant appearing in inequality (12) and for each $x \in E$, $Qx = \lim_{r \to 0} J_r x$. We now prove the following theorems.

### 3.1. *Convergence Theorems for m-Accretive Operators*

THEOREM 3. *Let $E$ be a real Banach space which is both uniformly convex and q-uniformly smooth. Let $T : D(T) \subseteq E \to E$ be a Lipschitz m-accretive operator with a closed domain $D(T)$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying*: (i) $0 \leq \alpha_n, \beta_n < 1$; $n \geq 0$; (ii) $\lim \alpha_n = 0$, $\lim \beta_n = 0$; (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then for any $x_0 \in D(T)$, the sequence $\{p_n\}_{n=0}^{\infty}$ in $E$ generated from $x_0$ by*

$$p_n = (1 - \alpha_n)x_n + \alpha_n(f - TQy_n), \qquad n \geq 0 \tag{14}$$

$$y_n = (1 - \beta_n)x_n + \beta_n(f - Tx_n), \qquad n \geq 0, \tag{15}$$

$$x_n = Qp_{n-1}, \qquad n \geq 1 \tag{16}$$

*converges strongly to the unique solution of the equation $x + Tx = f$.*

*Proof.* Since $T$ is $m$-accretive the equation $x + Tx = f$ has a solution $x^* \in D(T)$. Set $Sx = f - Tx$ and observe that $x^*$ is a fixed point of $S$ and that $S$ is Lipschitzian with constant $L$. Moreover, for each $x, y \in D(T)$, $\langle Sx - Xy, j(x - y)\rangle \leq 0$ so that $\langle Sx - Sy, j_q(x - y)\rangle \leq 0$. Starting with $x_0 \in D(T)$, we first compute $y_0 = (1 - \beta_0)x_0 + \beta_0(f - Tx_0)$ in $E$ and then compute $p_0 = (1 - \alpha_0)x_0 + \alpha_0(f - TQy_0)$ in $E$. We can now compute $x_1$ in $D(T)$ by $x_1 = Qp_0$. With $x_1$ we compute $y_1 = (1 - \beta_1)x_1 + \beta_1(f - Tx_1)$, then, $p_1 = (1 - \alpha_1)x_1 + \alpha_1(f - TQy_1)$ and also $x_2 = QP_1$. Continuing this process we generate the sequences $\{p_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$. Using (12) and (14)–(16) and the fact that $Q$ is a retraction, we obtain

$$\begin{aligned}
\|p_n - x^*\|^q &\\
&= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(SQy_n - x^*)\|^q \\
&\le (1 - \alpha_n)^q\|x_n - x^*\|^q + q\alpha_n(1 - \alpha_n)^{q-1}\langle SQy_n - x^*, j_q(x_n - x^*)\rangle \\
&\quad + c\alpha_n^q\|SQy_n - x^*\|^q
\end{aligned} \tag{17}$$

$$\begin{aligned}
\langle SQy_n - x^*, j_q(x_n - x^*)\rangle &\\
&= \langle SQy_n - Sx_n, j_q(x_n - x^*)\rangle + \langle Sx_n - x^*, j_q(x_n - x^*)\rangle \\
&\le \langle SQy_n - Sx_n, j_q(x_n - x^*)\rangle \le L\|Qy_n - x_n\| \cdot \|x_n - x^*\|^{q-1} \\
&\le L\|y_n - x_n\| \cdot \|x_n - x^*\|^{q-1} = L\beta_n\|x_n - Sx_n\| \cdot \|x_n - x^*\|^{q-1} \\
&\le L(1 + L)\beta_n\|x_n - x^*\|^q,
\end{aligned}$$

i.e.,

$$\langle SQy_n - x^*, j_q(x_n - x^*)\rangle \le L(1 + L)\beta_n\|x_n - x^*\|^q. \tag{18}$$

Also,

$$\begin{aligned}
\|SQy_n - x^*\|^q &\\
&\le L^q\|y_n - x^*\|^q \\
&\le L^q[(1 - \beta_n)^q\|x_n - x^*\|^q + q\beta_n(1 - \beta_n)^{q-1}\langle Sx_n - x^*, j_q(x_n - x^*)\rangle \\
&\quad + c\beta_n^q\|Sx_n - x^*\|^q] \\
&\le L^q[(1 - \beta_n)^q + c\beta_n^q L^q]\|x_n - x^*\|^q,
\end{aligned}$$

i.e.,

$$\|SQy_n - x^*\|^q \le L^q(1 - \beta_n)^q + c\beta_n^q L^q]\|x_n - x^*\|^q. \tag{19}$$

Using (18) and (19) in (17) we obtain

$$\begin{aligned}
\|p_n - x^*\|^q \le\ & [(1 - \alpha_n)^q + qL(1 + L)\alpha_n\beta_n(1 - \alpha_n)^{q-1} \\
&\quad + cL^q\alpha_n^q\{(1 - \beta_n)^q + c\beta_n^q L^q\}]\|x_n - x^*\|^q \\
\le\ & [(1 - \alpha_n)^q + qL(1 + L)\alpha_n\beta_n(1 - \alpha_n)^{q-1} \\
&\quad + cL^q\alpha_n^q(1 + cL^q)]\|x_n - x^*\|^q.
\end{aligned}$$

Conditions (ii) imply there exists $N_0 > 0$ such that for all $n \geq N_0$, $\alpha_n^{q-1} \leq [2qL^q c(1 + L^q c)]^{-1}$, $\beta_n \leq [2q^2 L(1 + L)]^{-1}$ so that

$$\|p_n - x^*\|^q \leq \left[(1 - \alpha_n)^q + \frac{1}{2q}\alpha_n(1 - \alpha_n)^{q-1} + \frac{1}{2q}\alpha_n\right]\|x_n - x^*\|^q$$
$$\leq \left[1 - \alpha_n + \frac{1}{q}\alpha_n\right]\|x_n - x^*\|^q = [1 - \alpha_n(1 - k)]\|x_n - x^*\|^q,$$
$$\text{where } k = \frac{1}{q} \in (0, 1).$$

Thus, for all $n \geq N_0$

$$\begin{aligned}\|p_n - x^*\|^q &\leq [1 - (1 - k)\alpha_n]\|p_{n-1} - x^*\|^q \\ &\leq \exp(-(1 - k)\alpha_n)\|p_{n-1} - x^*\|^q.\end{aligned} \tag{20}$$

Iteration of this inequality from $n = N_0 + 1$ to $N$ yields

$$\|p_N - x^*\|^q \leq \exp\left(-(1 - k)\sum_{j=N_0+1}^{N}\alpha_j\right)\|p_{N_0} - x^*\|^q \to 0$$

as $N \to \infty$ (by condition (iii)), completing the proof. ∎

*Error Estimate.* Observe that (20) implies

$$\|p_n - x^*\| \leq \|p_{n-1} - x^*\| \qquad \text{for all } n \geq N_0.$$

Moreover, since $\{\alpha_n\} \subseteq (0, 1)$ and $q > 1$, (20) also yields

$$\|p_n - x^*\|^q \leq [1 - (1 - k)^q\alpha_n^q]\|p_{n-1} - x^*\|^q, \tag{21}$$

for all $n \geq N_0$. Choose $\alpha_n = (1 - k)^{-1}[1 - (1 + n^q)/(1 + n)^q]^{1/q}$, $n \geq 0$. Then, clearly $\lim_{n\to\infty} \alpha_n = 0$. Furthermore,

$$\alpha_n^q \geq (1 - k)^{-q}\left[\frac{1}{1 + n} - \frac{1}{(1 + n)^q}\right], \qquad \text{so that } \sum_{n=0}^{\infty} \alpha_n^q = \infty \text{ (since } q > 1).$$

Hence $\sum_{n=0}^{\infty} \alpha_n = \infty$. Substitution of $\alpha_n$ in (21) now yields

$$\|p_n - x^*\|^q \leq \left[\frac{1 + n^q}{(1 + n)^q}\right]\|p_{n-1} - x^*\|^q,$$

i.e.,

$$(1 + n)^q \|p_n - x^*\|^q - n^q \|p_{n-1} - x^*\|^q \le \|p_{n-1} - x^*\|^q. \tag{22}$$

Summing this inequality from $n = N_0 + 1$ to $m$ ($m > N_0$) we obtain

$$(m + 1)^q \|p_m - x^*\|^q - (N_0 + 1)^q \|p_{N_0} - x^*\|^q \le (m - N_0) \|p_{N_0} - x^*\|^q,$$

i.e.,

$$\|p_m - x^*\| \le \left[\frac{m - N_0 + (N_0 + 1)^q}{m^q}\right]^{1/q} \|p_{N_0} - x^*)\|$$

so that $p_m \to x^*$ as $m \to \infty$; and clearly,

$$\|p_m - x^*\| = O(m^{-(q-1)/q}).$$

*Remark* 2. If $q = 2$, the error estimate of Theorem 3 becomes $\|x_n - x^*\| = O(n^{-1/2})$. Recall that $L_p$ spaces, $p \ge 2$, are 2-uniformly smooth. Thus, the error estimate of Theorem 3 agrees with the error estimate obtained in [8] for Hilbert spaces, and agrees with that obtained in [9] for $L_p$ spaces, $p \ge 2$. If $q = p$, the error estimate agrees with that obtained in [9] for $L_p$ spaces, $1 < p < 2$. Thus the error estimates obtained in [8], [9] are special cases of that obtained in Theorem 3.

COROLLARY 1. *Let $E$ and $T$ be as in Theorem* 3. *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence in* (0, 1) *satisfying the following conditions*: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then, for any $x_0 \in D(T)$, the sequence $\{p_n\}_{n=0}^{\infty}$ in $E$ generated from $x_0$ by*

$$p_n = (1 - \alpha_n)x_n + \alpha_n(f - Tx_n), \qquad n \ge 0,$$
$$x_n = Qp_{n-1}, \qquad n \ge 1$$

*converges strongly to the unique solution of the equation $x + Tx = f$.*

*Proof.* This follows from Theorem 3 with $\beta_n \equiv 0$ for each $n$. ∎

*Remark* 3. If, in Theorem 3, $D(T) = E$, the use of the projection operator $Q$ will not be necessary. Moreover, $E$ need not be uniformly convex. In particular, we have the following theorem.

THEOREM 4. *Let $E$ be a $q$-uniformly smooth Banach space and $T: D(T) = E \to E$ be a Lipschitz accretive operator. Let $\{\alpha_n\}$, $\{\beta_n\}$ be as in Theorem* 3. *Then given any $x_0 \in E$, the sequence $\{x_n\}$ generated from $x_0$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f - Ty_n), \qquad n \geq 0$$

$$y_n = (1 - \beta_n)x_n + \beta_n(f - Tx_n), \qquad n \geq 0,$$

*converges strongly to the unique solution of* $x + Tx = f$.

*Proof.* A result of Browder [2] shows that $T$ is $m$-accretive, and so the equation $x + Tx = f$ has a solution. The results follows as in the proof of Theorem 3. ∎

An immediate consequence of Theorem 4 is the following result:

COROLLARY 2. *Let E and T be as in Theorem 4. Let* $\{\alpha_n\}_{n=0}^{\infty}$ *be a real-sequence satisfying*: (i) $0 \leq \alpha_n < 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$; (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then for any* $x_0 \in E$, *the sequence* $\{x_n\}$ *generated from* $x_0$ *by* $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f - Tx_n)$, $n \geq 0$ *converges strongly to the unique solution of the equation* $x + Tx = f$.

*Proof.* This is obvious from Theorem 4. ∎

*Remark* 4. If the operator $T$ is assumed to have a *bounded range* the following convergence theorem is proved in real Banach spaces which are both uniformly convex and uniformly smooth.

THEOREM 5. *Let E be a real Banach space which is both uniformly convex and uniformly smooth. Let* $T : D(T) \subset E \to E$ *be an m-accretive operator with closed domain* $D(T)$, *and bounded range* $R(T)$. *Let* $\{c_n\}_{n=0}^{\infty}$ *in* $(0, 1)$ *be a real sequence satsifying*: (i) $\lim_{n\to\infty} c_n = 0$; (ii) $\sum_{n=0}^{\infty} c_n(1 - c_n) = \infty$, (iii) $\sum_{n=0}^{\infty} c_n b(c_n) < \infty$. *Then for any* $x_0 \in D(T)$, *the sequence* $\{p_n\}_{n=0}^{\infty}$ *in E generated from* $x_0$ *by* $p_n = (1 - c_n)x_n + c_n(f - Tx_n)$, $n \geq 0$; $x_n = Qp_{n-1}$, $n \geq 1$ *converges strongly to the unique solution of the equation* $x + Tx = f$.

*Proof.* The existence of a solution follows from the $m$-accretivity of the operator $T$. Let $x^*$ be a solution. Using inequality (13) we obtain

$$\begin{aligned}
\|p_n - x^*\|^2 &= \|(1 - c_n)(x_n - x^*) + c_n(Sx_n - x^*)\|^2, \\
&\qquad\qquad \text{where } Sx = f - Tx \text{ for each } x \in D(T) \\
&\leq (1 - c_n)^2\|x_n - x^*\|^2 + 2c_n(1 - c_n)\langle Sx_n - x^*, j(x_n - x^*)\rangle \\
&\quad + \max\{(1 - c_n)\|x_n - x^*\|, 1\}c_n\|Sx_n - x^*\|b(c_n\|Sx_n - x^*\|) \\
&\leq (1 - c_n)^2\|x_n - x^*\|^2 + \max\{(1 - c_n)\|x_n - x^*\|, 1\}c_n\|Sx_n \\
&\quad - x^*\|\max\{\|Sx_n - x^*\|, 1\}b(c_n) \\
&\leq (1 - c_n)^2\|x_n - x^*\|^2 + Mc_nb(c_n)\max\{(1 - c_n)\|x_n - x^*\|, 1\}
\end{aligned}$$

for some constant $M > 0$. We now consider the following two cases:

*Case* 1. We consider the set of all integers $n \geq 0$ for which

$$\max\{(1 - c_n)\|x_n - x^*\|, 1\} = 1.$$

In this case we obtain

$$\|p_n - x^*\|^2 \leq (1 - c_n)^2\|x_n - x^*\|^2 + Mc_nb(c_n).$$

*Case* 2. We consider the set of all integers $n \geq 0$ for which

$$\max\{(1 - c_n)\|x_n - x^*\|, 1\} = (1 - c_n)\|x_n - x^*\|.$$

In this case we obtain

$$\|p_n - x^*\|^2 \leq [(1 - c_n)^2 + Mc_n(1 - c_n)^2b(c_n)]\|x_n - x^*\|^2.$$

Thus, for all integers $n \geq 0$ we have

$$\begin{aligned}\|p_n - x^*\|^2 &\leq [(1 - c_n)^2 + Mc_n(1 - c_n)b(c_n)]\|x_n - x^*\|^2 + Mc_nb(c_n)\\ &\leq [(1 - c_n)^2 + Mc_n(1 - c_n)b(c_n)]\|p_{n-1} - x^*\|^2 + Mc_nb(c_n)\\ &= [1 - 2c_n + c_n^2 + Mc_n(1 - c_n)b(c_n)]\|p_{n-1} - x^*\|^2\\ &\quad + Mc_nb(c_n)\\ &= [1 - \{c_n(1 - c_n) + c_n[1 - M(1 - c_n)b(c_n)\}]\|p_{n-1} - x^*\|^2\\ &\quad + Mc_nb(c_n).\end{aligned}$$

Condition (i) and the continuity of (b) imply there exists some integer $N_0 > 0$ such that for all $n > N_0$

$$M(1 - c_n)b(c_n) \in (0, 1) \qquad \text{and} \qquad c_n(1 - c_n) + c_n[1 - M(1 - c_n)b(c_n)] \in (0, 1).$$

Let $\mu_n = c_n(1 - c_n) + c_n[1 - M(1 - c_n)b(c_n)]$, $n \geq N_0$. From the above calculations we obtain

$$\|p_n - x^*\|^2 \leq (1 - \mu_n)\|p_{n-1} - x^*\|^2 + Mc_nb(c_n), \qquad n \geq N_0.$$

Set $\rho_n = \|p_{n-1} - x^*\|^2$, $\sigma_n^2 = Mc_nb(c_n)$ so that the last inequality reduces to

$$\rho_{n+1} \leq (1 - \mu_n)\rho_n + \sigma_n^2. \tag{23}$$

Conditions (ii) and (iii) imply $\sum \mu_n = \infty$, $\sum \sigma_n^2 < \infty$. Inequality (23) and a simple induction now yield that for all $n \geq 1$,

$$0 \leq \rho_n \leq T^2 \alpha_n, \tag{24}$$

where $\alpha_n \geq 0$ is recursively generated by

$$\alpha_{n+1} = (1 - \mu_n)\alpha_n + \sigma_n^2, \qquad \alpha_1 = 1, \tag{25}$$

and $T^2 = \max\{\rho_1, 1\}$. It now follows from Theorem 2 that $\alpha_n \to 0$ as $n \to \infty$, so that from (24) we conclude that $\rho_n \to 0$ as $n \to \infty$, completing the proof of the theorem. ∎

THEOREM 6. *Let $E$ be a real uniformly smooth Banach space. Let $T : D(T) = E \to E$ be a continuous accretive operator with bounded range $R(T) \subseteq E$. Let $\{c_n\}_{n=0}^{\infty}$ be as in Theorem 5. Then, for any $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated from $x_0$ by*

$$x_{n+1} = (1 - x_n)x_n + c_n(f - Tx_n), \qquad n \geq 0,$$

*converges strongly to the unique solution of the equation $x + Tx = f$.*

*Proof.* The existence of a solution follows from Martin [18]. Following the technique of the proof of Theorem 5 we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq [1 - \{c_n(1 - c_n) \\ + c_n[1 - M(1 - c_n(b(c_n)]\}]\|x_n - x^*\|^2 + Mc_n b(c_n),\end{aligned}$$

and the rest of the argument follows exactly as in the proof of Theorem 3 to yield

$$\rho_{n+1} \leq (1 - \mu_n)\rho_n + \sigma_n^2, \qquad \text{where } \rho_n = \|x_{n-1} - x^*\|^2$$

The result follows as in Theorem 5. ∎

### 3.2. *Convergence Theorems for Dissipative Operators*

In this section we turn our attention to convergence theorems for dissipative operators. We shall be interested in the approximation of a solution of the equation $x - \lambda Tx = f$, where $T : D(T) \subseteq E \to E$ is $m$-dissipative and $\lambda$ is a real positive constant. In particular, we prove the following theorems.

THEOREM 7. *Let $E$ be a real Banach space which is both uniformly convex and $q$-uniformly smooth. Let $T : D(T) \subseteq E \to E$ be a Lipschitz m-*

*dissipative operator with a closed domain $D(T)$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying the following conditions:* (I) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\lim \beta_n = 0$; (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then for any $x_0 \in D(T)$, the sequence $\{p_n\}_{n=0}^{\infty}$ in $E$ generated from $x_0$ by*

$$p_n = (1 - \alpha_n)x_n + \alpha_n(f + \lambda TQy_n), \qquad n \geq 0$$
$$y_n = (1 - \beta_n)x_n + \beta_n(f + \lambda Tx_n), \qquad n \geq 0,$$
$$x_n = Qp_{n-1}, \qquad n \geq 1$$

*converges strongly to the unique solution of the equation $x - \lambda Tx = f$, $\lambda > 0$.*

*Proof.* The existence of a solution follows from the $m$-dissipativity of $T$. Furthermore $(-\lambda T)$ is Lipschitz and $m$-accretive. The result now follows from Theorem 3. ∎

COROLLARY 3. *Let $E$ and $T$ be as in Theorem 7. Let $\{\alpha_n\}$ be a real sequence in* (0, 1) *satisfying the following conditions:* (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then, for any $x_0 \in D(T)$, the sequence $\{p_n\}_{n=0}^{\infty}$ in $E$ generated from $x_0$ by*

$$p_n = (1 - \alpha_n)x_n + \alpha_n(f + \lambda Tx_n), \qquad n \geq 0$$
$$x_n = Qp_{n-1}, \qquad n \geq 1$$

*converges strongly to the unique solution of the equation $x - \lambda Tx = f$.*

*Proof.* This is obvious. Set $\beta_n \equiv 0$ for all $n$ in Theorem 7. ∎

*Remark* 5. Following the pattern of Theorem 7 and Corollary 3, all the results of Subsection 3.1 can be restated in terms of dissipative operators for the equation $x - \lambda Tx = f$.

For example, Theorem 4 and Corollary 2 can be stated for dissipative operators as follows:

THEOREM 8. *Let $E$ be a $q$-uniformly smooth Banach space and $T : D(T) = E \to E$ be a Lipschitz dissipative operator. Let $\{\alpha_n\}$, $\{\beta_n\}$ be as in Theorem 4. Then given any $x_0 \in E$, the sequence $\{x_n\}$ generated from $x_0$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + \lambda Ty_n), \qquad n \geq 0,$$
$$y_n = (1 - \beta_n)x_n + \beta_n(f + \lambda Tx_n), \qquad n \geq 0$$

*converges strongly to the unique solution of $x - \lambda Tx = f$, $\lambda > 0$.*

COROLLARY 4. *Let E and T be as in Theorem 8. Let* $\{\alpha_n\}_{n=0}^{\infty}$ *be a real sequence satisfying the following conditions*: (i) $0 \le \alpha_n < 1$, (ii) $\lim \alpha_n = 0$, (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then for any* $x_0 \in E$, *the sequence* $\{x_n\}_{n=1}^{\infty}$ *generated from* $x_0$ *by* $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + \lambda Tx_n)$, $n \ge 0$ *converges strongly to the unique solution of* $x - \lambda Tx = f$.

## ACKNOWLEDGMENTS

One of the authors (C.E.C.) is grateful to Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

## REFERENCES

1. V. BARBU, "Nonlinear Semigroups and Differential Equations in Banach Spaces," Noordhoff, Leyden, The Netherlands, 1976.
2. F. E. BROWDER, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, *Bull. Amer. Math. Soc.* **73** (1967), 875–882.
3. F. E. BROWDER, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *in* "Proc. Sympos. Pure Math.," Vol. 18, Part 2, Amer. Math. Soc., Providence RI, 1976.
4. F. E. BROWDER, Nonlinear monotone and accretive operators in Banach spaces, *Proc. Nat. Acad. Sci. U.S.A.* **61** (1968), 388–393.
5. F. E. BROWDER AND W. V. PETRYSHYN, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.* **20** (1967), 197–228.
6. R. E. BRUCK, The iterative solution of the equation $y \in x + Tx$ for a monotone operator $T$ in Hilbert spaces, *Bull. Amer. Math. Soc.* **79** (1973), 1258–1261.
7. C. E. CHIDUME, The iterative solution of the equation $y \in x + Tx$ for a monotone operator $T$ in $L^p$ spaces, *J. Math. Anal. Appl.* **116** (1986), 531–537.
8. C. E. CHIDUME, An approximation method for monotone Lipschitzian operators in Hilbert spaces, *J. Austral. Math. Soc. Ser. A* **41** (1986), 59–63.
9. C. E. CHIDUME, Iterative solution of nonlinear equations of the monotone type in Banach spaces, *Bull. Austral. Math. Soc.* **42** (1990), 21–321.
10. C. E. CHIDUME, Iterative solution of nonlinear equations of the monotone and dissipative type, *Appl. Anal.* **33** (1989), 79–86.
11. I. CIORENESCU, "Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems," Kluwer Academic Publishers, 1990.
12. K. DEIMLING, Zeros of accretive operators, *Manuscripta Math.* **13** (1974), 365–374.
13. W. G. DOTSON, An iterative process for nonlinear monotone nonexpansive operators in Hilbert spaces, *Math. Comp.* **32** (1978), 223–225.
14. J. C. DUNN, Iterative construction of fixed points for multivalued operators of the monotone type, *J. Funct. Anal.* **27** (1978), 38–50.
15. S. ISHIKAWA, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.* **149** (1974), 147–150.
16. T. KATO, Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan* **19** (1967), 508–520.
17. W. R. MANN, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953), 506–510.

18. R. H. Martin, A global existence theorem for autonomous differential equations in Banach spaces, *Proc. Amer. Math. Soc.* **26** (1970), 307–314.
19. G. J. Minty, Monotone (nonlinear) operators in Hilbert spaces, *Duke Math. J.* **29** (1962), 541–546.
20. O. Nevanlinna, Global iteration schemes for monotone operators, *Nonlinear Anal.* **3,** No. 4 (1979).
21. O. Nevanlinna and S. Reich, Strong convergence of contraction semigroups and of iteration methods for accretive operators in Banach spaces, *Israel J. Math.* **32** (1979), 44–58.
22. M. O. Osilike, Ishikawa and Mann iteration methods for nonlinear strongly accretive mappings, *Bull. Austral. Math. Soc.* **46** (1992), 411–422.
23. S. Reich, An iterative procedure for constructing zeros of accretive sets in Banach spaces, *Nonlinear Anal.* **2** (1978), 85–92.
24. S. Reich, Constructive techniques for accretive and monotone operators, *in* "Applied Nonlinear Analysis," pp. 335–345, Academic Press, New York, 1979.
25. S. Reich, Constructing zeros of accretive operators, I, II, *Appl. Anal.* **8** (1974), 349–352; *Appl. Anal.* **9** (1979), 159–163.
26. S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.* **75** (1980), 287–292.
27. B. E. Rhoades, Comments on two fixed point iteration methods, *J. Math. Anal. Appl.* **56** (1976), 741–750.
28. V. L. Smul'yan, On the derivative of the norm of Banach spaces, *Dokl. Akad. Nauk SSSR* **27** (1940), 255–258.
29. M. M. Vainverg, "Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations," Wiley, New York, 1973.
30. Hong-Kun Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.* **16,** No. 12 (1991), 1127–1138.
31. Zongben Xu and G. F. Roach, Characteristic inequalities for uniformly convex and uniformly smooth Banach spaces, *J. Math. Anal. Appl.* **157** (1991), 189–210.
32. Zongben Xu, Bo Zhang, and G. F. Roach, On the steepest descent approximation to solutions of nonlinear strongly accretive operator equations, *J. Comput. Math.* **7,** No. 2 (1992).
33. E. H. Zarantonello, "Solving Functional Equations by Contractive Averaging," Tech. Rep. No. 160, U.S. Army Math. Res. Centre, Madison, WI, 1960.
34. E. Zeidler, "Nonlinear Functional Analysis and Its Applications. Part II. Monotone Operators," Springer-Verlag, New York/Berlin, 1985.