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Ishikawa and Mann Iteration Methods with Errors for Nonlinear Equations of the Accretive Type*

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Let E be an *arbitrary Banach space* and $T\colon E \to E$ a Lipschitz strongly accretive operator. It is proved that for a given $f \in E$, the Ishikawa and the Mann iteration methods with errors introduced by L.-S. Liu (*J. Math. Anal. Appl.* **194,** 1995, 114–125) converge strongly to the solution of the equation $Tx = f$. Furthermore, if E is a uniformly smooth Banach space and $T\colon E \to E$ is demicontinuous and strongly accretive, it is also proved that both the Ishikawa and the Mann iteration methods with errors converge strongly to the solution of the equation $Tx = f$. Related results deal with the iterative approximation of fixed points of strongly pseudocontractive operators, and the solution of the equation $x + Tx = f$, $f \in E$ when $T\colon E \to E$ is m-accretive. © 1997 Academic Press

1. INTRODUCTION

Suppose E is an arbitrary Banach space. We denote by j the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^*\colon \operatorname{Re}\langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued. In the sequel we shall denote the single-valued normalized duality mapping by j.

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An operator T with domain $D(T)$ and range $R(T)$ in E is called *strongly accretive* if for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a constant $k > 0$ such that

$$\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2. \tag{1}$$

Without loss of generality we may assume $k \in (0,1)$. If $k = 0$ in (1) then T is called *accretive*. If T is accretive and $(I + rT)(D(T)) = E$ for all $r > 0$ then T is called *m-accretive*.

Closely related to the class of strongly accretive operators is the class of *strongly pseudocontractive operators* where an operator T is called a *strong pseudocontraction* if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a constant $t > 1$ such that

$$\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \leq \frac{1}{t}\|x-y\|^2. \tag{2}$$

If I denotes the identity operator, it follows from inequalities (1) and (2) that T is strongly accretive if and only if $(I - T)$ is strongly pseudocontractive. Thus the mapping theory of strongly accretive operators is closely related to the fixed point theory of strongly pseudocontractive operators. Recent interest in mapping theory of strongly accretive operators, particularly as it relates to existence theorems for nonlinear ordinary and partial differential equations, has prompted a corresponding interest in fixed point theory of strong pseudocontractions.

It is well known (see, for example, Theorem 13.1 of Deimling [11]) that for any given $f \in E$ the equation

$$Tx = f \tag{3}$$

has a unique solution if $T\colon E \to E$ is strongly accretive and continuous, or E is uniformly smooth and $T\colon E \to E$ is strongly accretive and demicontinuous. Martin [20] has also proved that if $T\colon E \to E$ is continuous and accretive then T is m-accretive so that for any given $f \in E$ the equation

$$x + Tx = f \tag{4}$$

has a unique solution.

Several authors have applied the *Mann iteration method* [19] and the *Ishikawa iteration method* [16] to approximate solutions (when they exist) of Eqs. (3) and (4) and fixed points of strong pseudocontractions with nonempty fixed-point sets (see, for example, [2–9, 12–15, 23–25]).

Recently, Liu [18] introduced the following iteration methods which he called *Ishikawa and Mann iteration methods with errors*.

(a) *Ishikawa Iteration Method with Errors* [18, p. 116]. For a nonempty set K of E and a mapping $T\colon K \to E$, the sequence $\{x_n\}_{n=0}^{\infty} \subseteq K$ is defined for arbitrary $x_0 \in K$ by

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n + u_n, \qquad n \geq 0$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n + v_n, \qquad n \geq 0,$$

where $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two summable sequences in E (i.e., $\sum_{n=0}^{\infty}\|u_n\| < \infty$ and $\sum_{n=0}^{\infty}\|v_n\| < \infty$), $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are two real sequences in $[0, 1]$ satisfying suitable conditions.

(b) *Mann Iteration Method with Errors* [18, p. 116]. With K, T, and x_0 as in (a) the sequence $\{x_n\}$ in K is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n + u_n, \qquad n \geq 0,$$

where $\{u_n\}$ is a summable sequence in E and $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying suitable conditions.

If $u_n \equiv 0$, $v_n \equiv 0$ then the Ishikawa and Mann iteration methods with errors reduce to the original Ishikawa and Mann iteration methods.

In his study of Ishikawa and Mann iteration methods with errors for strongly accretive operators, Liu [18] proved the following theorems:

THEOREM L1 [18, p. 119]. *Let E be a uniformly smooth Banach space and $T\colon E \to E$ a Lipschitz strongly accretive operator with constant $k \in (0,1)$ and Lipschitz constant $L \geq 1$. Define $S\colon E \to E$ by $Sx = f + x - Tx$. Let $\{u_n\}$ and $\{v_n\}$ be two summable sequences in E and let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in* $[0, 1]$ *satisfying the condition* (i) $\lim_{n\to\infty}\alpha_n = 0$, (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$, *and* (iii) $\limsup_{n\to\infty}\beta_n < k/(L^2 - k)$. *For arbitrary $x_0 \in E$, define the sequence $\{x_n\}$ by*

$$y_n = (1 - \beta_n)x_n + \beta_n Sx_n + u_n, \qquad n \geq 0$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sy_n + v_n, \qquad n \geq 0.$$

Suppose $\{Sy_n\}$ is bounded. Then the sequence $\{x_n\}$ converges strongly to the solution of the equation $Tx = f$.

THEOREM L2 [18, p. 123]. *Let E be a uniformly smooth Banach space and $T\colon E \to E$ a demicontinuous strongly accretive operator. Define $S\colon E \to E$ by $Sx = f + x - Tx$ and let $\{u_n\}$ be a summable sequence in E and $\{\alpha_n\}$ a real sequence in* $[0, 1]$ *satisfying the conditions* (i) $\lim \alpha_n = 0$, (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$. *For arbitrary $x_0 \in E$ define the sequence $\{x_n\}$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n + u_n, \qquad n \geq 0.$$

Suppose $\{Sx_n\}$ *is bounded. Then the sequence* $\{x_n\}$ *converges strongly to the solution of the equation* $Tx = f$.

Theorems similar to Theorems L1 and L2 are also proved in [18] for approximation of fixed points of strongly pseudocontractive operators.

It is our purpose in this paper to extend Theorem L1 to *arbitrary Banach spaces*, and without the boundedness assumption imposed on $\{Sy_n\}$. Also we extend Theorem L2 to the Ishikawa iteration method with errors. Our iteration parameters $\{\alpha_n\}$ and $\{\beta_n\}$ will be such that Theorem L2 will be a special case of our result. Furthermore, if E is an arbitrary Banach space and $T: E \to E$ is strongly accretive and *uniformly continuous*, we prove that the Ishikawa and the Mann iteration methods with errors converge strongly to the solution of the equation $Tx = f$ if $\{Sx_n\}$ and $\{Sy_n\}$ are bounded. We also prove similar convergence results for the equation $x + Tx = f$ when T is m-accretive, and for fixed points of strongly pseudo-contractive mappings with nonempty fixed-point sets.

We shall need the following result

THEOREM R [22, p. 89]. *Let* E *be a uniformly smooth Banach space. Then there exists a continuous nondecreasing function* $b: [0, \infty) \to [0, \infty)$ *such that* $b(0) = 0$, $b(ct) \le cb(t)\ \forall c \ge 1$ *and for all* $x, y \in E$,

$$\|x + y\|^2 \le \|x\|^2 + 2\operatorname{Re}\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|). \tag{5}$$

2. MAIN RESULTS

In the sequel $k \in (0, 1)$ is the constant appearing in the definition of strongly accretive operators and L is the Lipschitz constant of T.

THEOREM 1. *Let* E *be an arbitrary Banach space and* $T: E \to E$ *a Lipschitz strongly accretive operator. Let* $\{u_n\}$ *and* $\{v_n\}$ *be two summable sequences in* E *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *two real sequences in* $[0, 1]$ *satisfying the conditions* (i) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$, (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then for any given* $f \in E$ *the sequence* $\{x_n\}$ *defined for arbitrary* $x_0 \in E$ *by*

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n(f + (I - T)x_n) + u_n, \qquad n \ge 0 \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n(f + (I - T)y_n) + v_n, \qquad n \ge 0 \end{aligned} \tag{6}$$

converges strongly to the solution of the equation $Tx = f$.

Proof. The existence of a solution to the equation $Tx = f$ follows from [1, 10] and the uniqueness follows from the strong accretivity of T. Let x^* denote the unique solution. Define $S: E \to E$ by $Sx = f + (I - T)x$.

Then x^* is a fixed point of S and S is Lipschitz with constant $L_* = 1 + L$. Moreover, for all $x, y \in E$ there exists $j(x - y) \in J(x - y)$ such that

$$\mathrm{Re}\langle (I - S)x - (I - S)y, j(x - y)\rangle \geq k\|x - y\|^2.$$

Thus,

$$\mathrm{Re}\langle (I - S - kI)x - (I - S - kI)y, j(x - y)\rangle \geq 0,$$

and it follows from Lemma 1.1 of Kato [17] that for all $x, y \in E$ and $r > 0$ the inequality

$$\|x - y\| \leq \|x - y + r[(I - S - kI)x - (I - S - kI)y]\| \tag{7}$$

holds. From (6) we obtain

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n S y_n - v_n \\ &= (1 + \alpha_n)x_{n+1} + \alpha_n(I - S - kI)x_{n+1} - (1 - k)\alpha_n x_n \\ &\quad + (2 - k)\alpha_n^2(x_n - Sy_n) \\ &\quad + \alpha_n(Sx_{n+1} - Sy_n) - (1 + (2 - k)\alpha_n)v_n. \end{aligned}$$

Observe that

$$x^* = (1 + \alpha_n)x^* + \alpha_n(I - S - kI)x^* - (1 - k)\alpha_n x^*,$$

so that

$$\begin{aligned} &\|x_n - x^*\| \\ &\quad \geq (1 + \alpha_n)\|x_{n+1} - x^* \\ &\qquad + \frac{\alpha_n}{1 + \alpha_n}[(I - S - kI)x_{n+1} - (I - S - kI)x^*]\| \\ &\qquad - (1 - k)\alpha_n\|x_n - x^*\| - (2 - k)\alpha_n^2\|x_n - Sy_n\| \\ &\qquad - \alpha_n\|Sx_{n+1} - Sy_n\| \\ &\qquad - [1 + (2 - k)\alpha_n]\|v_n\| \\ &\quad \geq (1 + \alpha_n)\|x_{n+1} - x^*\| - (1 - k)\alpha_n\|x_n - x^*\| - (2 - k)\alpha_n^2\|x_n - Sy_n\| \\ &\qquad - \alpha_n\|Sx_{n+1} - Sy_n\| - [1 + (2 - k)\alpha_n]\|v_n\| \qquad (\text{using } (7)). \end{aligned}$$

Hence

$$\|x_{n+1} - x^*\| \leq \frac{[1 + (1-k)\alpha_n]}{1 + \alpha_n}\|x_n - x^*\| + (2-k)\alpha_n^2\|x_n - Sy_n\| + \alpha_n\|Sx_{n+1} - Sy_n\| + [1 + (2-k)\alpha_n]\|v_n\|. \quad (8)$$

Furthermore we have the estimates

$$\begin{aligned}\|y_n - x^*\| &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|Sx_n - x^*\| + \|u_n\| \\ &\leq [1 + \beta_n(L_* - 1)]\|x_n - x^*\| + \|u_n\| \\ &\leq L_*\|x_n - x^*\| + \|u_n\|, \\ \|x_n - Sy_n\| &\leq \|x_n - x^*\| + L_*\|y_n - x^*\| \\ &\leq [1 + L_*^2]\|x_n - x^*\| + L_*\|u_n\|, \qquad (9) \\ \|Sx_{n+1} - Sy_{n+1}\| &\leq L_*\|x_{n+1} - y_n\| \\ &\leq L_*[(1-\alpha_n)\|x_n - y_n\| + \alpha_n\|Sy_n - y_n\| + \|v_n\|] \\ &\leq L_*[(1-\alpha_n)\beta_n\|x_n - Sx_n\| + (1-\alpha_n)\|u_n\| \\ &\qquad + \alpha_n(1 + L_*)\|y_n - x^*\| + \|v_n\|] \\ &\leq [L_*(1+L_*)\beta_n + \alpha_n L_*^2(1 + L_*)]\|x_n - x^*\| \\ &\qquad + [\alpha_n L_*(1 + L_*) + L_*]\|u_n\| + L_*\|v_n\|. \qquad (10)\end{aligned}$$

Using (9) and (10) in (8) we obtain

$$\begin{aligned}&\|x_{n+1} - x^*\| \\ &\quad\leq \frac{[1 + (1-k)\alpha_n]}{1 + \alpha_n}\|x_n - x^*\| + (2-k)\alpha_n^2(1 + L_*^2)\|x_n - x^*\| \\ &\qquad + (2-k)\alpha_n^2 L_*\|u_n\| + \alpha_n[L_*(1 + L_*)\beta_n \\ &\qquad\qquad + \alpha_n L_*^2(1 + L_*)]\|x_n - x^*\| \\ &\qquad + \alpha_n[\alpha_n L_*(1 + L_*) + L_*]\|u_n\| + \alpha_n L_*\|v_n\| \\ &\qquad + [1 + (2-k)\alpha_n]\|v_n\| \\ &\quad\leq [1 + (1-k)\alpha_n][1 - \alpha_n + \alpha_n^2]\|x_n - x^*\| \\ &\qquad + [((2-k)(1 + L_*^2) + L_*^2(1 + L_*))\alpha_n^2 \\ &\qquad\qquad + L_*(1 + L_*)\alpha_n\beta_n]\|x_n - x^*\| \\ &\qquad + L_*(4 + L_*)\|u_n\| + (3 + L_*)\|v_n\|\end{aligned}$$

$$\leq \left[1 - \alpha_n k + \alpha_n^2\right] \|x_n - x^*\| + \left[\left((2-k)(1+L_*{}^2) + L_*{}^2(1+L_*)\right)\alpha_n^2 + L_*(1+L_*)\,\alpha_n\,\beta_n\right] \|x_n - x^*\| + L_*(4+L_*)\|u_n\| + (3+L_*)\|v_n\|$$
$$\leq [1 - \alpha_n k]\|x_n - x^*\| + \alpha_n\left[\left(L_*{}^3 + 3L_*{}^2 + 3\right)\alpha_n + L_*(1+L_*)\,\beta_n\right]\|x_n - x^*\| + L_*(4+L_*)\|u_n\| + (3+L_*)\|v_n\|.$$

Since $\lim[(L_*{}^3 + 3L_*{}^2 + 3)\alpha_n + L_*(1+L_*)\beta_n] = 0$, there exists a positive integer N such that $[(L_*{}^3 + 3L_*{}^2 + 3)\alpha_n + L_*(1+L_*)\beta_n] \leq k(1-k)$ for all $n \geq N$. Thus

$$\|x_{n+1} - x^*\| \leq \left[1 - \alpha_n k^2\right]\|x_n - x^*\| + L_*(4+L_*)\|u_n\| + (3+L_*)\|v_n\| \quad \forall n \geq N.$$

Set $\rho_n := \|x_n - x^*\|$, $\delta_n := \alpha_n k^2$, $\sigma_n^2 := L_*(4+L_*)\|u_n\| + (3+L_*)\|v_n\|$. Then $\rho_{n+1} \leq [1-\delta_n]\rho_n + \sigma_n^2$ for all $n \geq N$. Clearly, $\sum\delta_n = \infty$, and $\sum\sigma_n^2 < \infty$. Hence it follows as in [7] that $\lim \rho_n = 0$, completing the proof of Theorem 1.

Remark 1. Theorem 1 extends Theorem L1 and a host of other theorems (see, for example, Theorem 1 of Chidume [3], Theorem 2 of Chidume [5], Theorems 1 and 3 of Chidume and the author [8], Theorems 3.1 and 4.1 of Tan and Xu [25], Theorem 1 of Deng [12, 14], Theorems 1 and 3 of Deng [13], and Theorem 2 of Deng and Ding [15]) from Banach Spaces which are either uniformly convex or uniformly smooth to arbitrary Banach spaces. Moreover, the boundedness requirement imposed on $\{Sy_n\}$ in Theorem L1 is not imposed in Theorem 1 of our result.

If E is a uniformly smooth Banach space we prove the following theorem:

THEOREM 2. *Suppose E is a uniformly smooth Banach space and T: $E \to E$ a strongly accretive operator. Suppose $Tx = f$ has a solution and suppose S, $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{x_n\}$ are as in Theorem* 1. *Suppose the sequences $\{Sy_n\}$ and $\{Sx_n\}$ are bounded. Then the sequence $\{x_n\}$ converges strongly to the solution of the equation $Tx = f$.*

Proof. The strong accretivity of T implies that the solution of the equation is unique. Let x^* denote the unique solution of the equation. For all $x, y \in E$ there exists $j(x-y) \in J(x-y)$ such that

$$\mathrm{Re}\langle Sx - Sy, j(x-y)\rangle \leq (1-k)\|x-y\|^2. \tag{11}$$

Using (5) and (11) we obtain the estimates

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*)\|^2 \\
&\qquad + 2\operatorname{Re}\langle v_n, j\big((1-\alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*)\big)\rangle \\
&\qquad + \max\big\{\|(1-\alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*)\|, 1\big\}\|v_n\| b\big(\|v_n\|\big).
\end{aligned}$$

Let $d := \sup\{\|Sx_n - x^*\| + \|Sy_n - x^*\| : n \ge 0\} + \|x_0 - x^*\|$. Then by a simple induction we obtain

$$\|x_n - x^*\| \le d + \sum_{k=0}^{n-1} \|v_k\| \le d + \sum_{n=0}^{\infty} \|v_n\|, \qquad \forall n \ge 0.$$

Hence $\{x_n\}$ is bounded, and

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le {} & \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*)\|^2 \\
& + D_1\|v_n\| + D_2\|v_n\| b\big(\|v_n\|\big),
\end{aligned}$$

for some constants D_1 and D_2, since $\{x_n - x^*\}$ and $\{Sy_n - x^*\}$ are bounded. Set $\omega_n := \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*)\|^2$. Then

$$\begin{aligned}
\omega_n \le {} & (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\operatorname{Re}\langle Sy_n - x^*, j(x_n - x^*)\rangle \\
& + \max\big\{(1-\alpha_n)\|x_n - x^*\|, 1\big\}\alpha_n\|Sy_n - x^*\| b\big(\alpha_n\|Sy_n - x^*\|\big) \\
\le {} & (1-\alpha_n)^2\|x_n - x^*\| + 2\alpha_n(1-\alpha_n)(1-k)\|y_n - x^*\|^2 \\
& + 2\alpha_n(1-\alpha_n)\operatorname{Re}\langle Sy_n - x^*, j(x_n - x^*) - j(y_n - x^*)\rangle \\
& + D_3\alpha_n b(\alpha_n).
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le {} & (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)(1-k)\|y_n - x^*\|^2 \\
& + 2\alpha_n(1-\alpha_n)\operatorname{Re}\langle Sy_n - x^*, j(x_n - x^*) - j(y_n - x^*)\rangle \\
& + D_3\alpha_n b(\alpha_n) + D_1\|v_n\| + D_2\|v_n\| b\big(\|v_n\|\big). \qquad (12)
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&\|y_n - x^*\|^2 \\
&\quad \le \|(1-\beta_n)(x_n - x^*) + \beta_n(Sx_n - x^*)\|^2 \\
&\qquad + 2\operatorname{Re}\langle u_n, j((1-\beta_n)(x_n - x^*) + \beta_n(Sx_n - x^*))\rangle \\
&\qquad + \max\{\|(1-\beta_n)(x_n - x^*) + \beta_n(Sx_n - x^*)\|, 1\}\|u_n\| b(\|u_n\|) \\
&\quad \le \left[(1-\beta_n)^2 + 2\beta_n(1-\beta_n)(1-k)\right]\|x_n - x^*\|^2 \\
&\qquad + D_4\beta_n b(\beta_n) + D_5\|u_n\| + D_6\|u_n\| b(\|u_n\|) \\
&\quad \le (1-\beta_n k)^2\|x_n - x^*\|^2 + D_4\beta_n b(\beta_n) + D_5\|u_n\| + D_6\|u_n\| b(\|u_n\|)
\end{aligned}
$$

for some constants D_4, D_5, D_6. Let $D := \max\{D_i : 1 \le i \le 6\}$. Then

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \left[(1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)(1-k)(1-\beta_n k)^2\right]\|x_n - x^*\|^2 \\
&\qquad + 2D\alpha_n(1-\alpha_n)(1-k)\left[\beta_n b(\beta_n) + \|u_n\|(1 + b(\|u_n\|))\right] \\
&\qquad + 2\alpha_n(1-\alpha_n)\|Sy_n - x^*\|\,\|j(x_n - x^*) - j(y_n - x^*)\| \\
&\qquad + D\left[\alpha_n b(\alpha_n) + \|v_n\|(1 + b(\|v_n\|))\right] \\
&\quad \le [1 - \alpha_n k]\|x_n - x^*\|^2 + \alpha_n\Big[2D\big(\beta_n b(\beta_n) \\
&\qquad\qquad + \|u_n\|(1 + b(\|u_n\|))\big) + Db(\alpha_n) \\
&\qquad + 2\|Sy_n - x^*\|\,\|j(x_n - x^*) - j(y_n - x^*)\|\Big] + D\|v_n\|(1 + b(\|v_n\|)). \qquad (13)
\end{aligned}
$$

Observe that

$$\|(x_n - x^*) - (y_n - x^*)\| \le \beta_n\|x_n - Sx_n\| + \|u_n\| \to 0 \qquad \text{as } n \to \infty,$$

since $\lim \beta_n = 0$, $\lim\|u_n\| = 0$, and $\{x_n - Sx_n\}$ is bounded. Since $\{Sy_n - x^*\}$, $\{x_n - x^*\}$, $\{y_n - x^*\}$ are bounded subsets of E, by the uniform continuity of j on bounded subsets of E, we obtain $\lim\|Sy_n - x^*\|\,\|j(x_n - x^*) - j(y_n - x^*)\| = 0$. Since $\{v_n\}$ is bounded and b is nondecreasing, then $b(\|v_n\|)$ is bounded.

Set $\rho_n := \|x_n - x^*\|^2$, $\delta_n := \alpha_n k$,

$$
\begin{aligned}
\sigma_n := \alpha_n\Big[&2D\big(\beta_n b(\beta_n) + \|u_n\|(1 + \|u_n\|)\big) + Db(\alpha_n) \\
&+ 2\|Sy_n - x^*\|\,\|j(x_n - x^*) - j(y_n - x^*)\|\Big],
\end{aligned}
$$

and $\lambda_n := D\|v_n\|(1 + b(\|v_n\|))$. Then

$$\rho_{n+1} \leq [1 - \delta_n]\,\rho_n + \sigma_n + \lambda_n \qquad \forall n \geq 0.$$

Clearly $\sum \delta_n = \infty$, $\sigma_n = o(\delta_n)$, and $\sum \lambda_n < \infty$. Hence it follows from Lemma 2 of [18] that $\lim \rho_n = 0$, completing the proof of Theorem 2.

COROLLARY 1. *Suppose E is a uniformly smooth Banach space and $T: E \to E$ is a demicontinuous strongly accretive operator. Suppose S, $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{x_n\}$ are as in Theorem 2. Suppose $\{Sy_n\}$ and $\{Sx_n\}$ are bounded. Then the sequence $\{x_n\}$ converges strongly to the solution of the equation $Tx = f$.*

Proof. The existence of a solution follows from Deimling [11] and the result follows from Theorem 2.

Remark 2. Theorem L2 is a special case of Corollary 1 for which $\beta_n \equiv 0$.

If the range of $(I - T)$ is bounded, then $\{Sx_n\}$ and $\{Sy_n\}$ are bounded, hence Theorem 1 of a recent result of Chidume [5] is also a special case of Corollary 1 of our result for which $\beta_n \equiv 0$.

If we retain the hypothesis that $\{Sx_n\}$ and $\{Sy_n\}$ are bounded then we obtain the following theorem in arbitrary Banach spaces.

THEOREM 3. *Let E be an arbitrary Banach space and let $T: E \to E$ be a uniformly continuous strongly accretive operator. Let $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{x_n\}$, $\{Sx_n\}$, and $\{Sy_n\}$ be as in Theorem 2. Then the sequence $\{x_n\}$ converges strongly to the solution of the equation $Tx = f$.*

Proof. The existence of a solution follows from Deimling [10]. Let x^* denote the solution. Then as in the proof of Theorem 1 we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \frac{[1 + (1 - k)\alpha_n]}{1 + \alpha_n}\|x_n - x^*\| + (2 - k)\alpha_n^2\|x_n - Sy_n\| \\
&\quad + \alpha_n\|Sx_{n+1} - Sy_n\| + [1 + (2 - k)\alpha_n]\|v_n\| \\
&\leq [1 - \alpha_n k + \alpha_n^2]\|x_n - x^*\| + (2 - k)\alpha_n^2\|x_n - Sy_n\| \\
&\quad + \alpha_n\|Sx_{n+1} - Sy_n\| + [1 + (2 - k)\alpha_n]\|v_n\|.
\end{aligned}$$

It follows as in the proof of Theorem 2 that

$$\|x_n - x^*\| \le d + \sum_{n=0}^{\infty} \|v_n\| = M \qquad \text{and}$$

$$\|x_n - Sy_n\| \le \|x_n - x^*\| + \|Sy_n - x^*\| \le 2M,$$

where $d = \sup\{\|Sx_n - x^*\| + \|Sy_n - x^*\|\colon n \ge 0\} + \|x_0 - x^*\|$. Hence

$$\|x_{n+1} - x^*\| \le [1 - \alpha_n k]\|x_n - x^*\| + \alpha_n[5M\alpha_n + \|Sx_{n+1} - Sy_n\|] + 3\|v_n\|. \tag{14}$$

Observe that

$$\begin{aligned}
\|x_{n+1} - y_n\| &\le (1 - \alpha_n)\|x_n - y_n\| + \alpha_n\|Sy_n - y_n\| + \|v_n\| \\
&\le (1 - \alpha_n)\beta_n\|x_n - Sx_n\| + (1 - \alpha_n)\|u_n\| \\
&\qquad + \alpha_n[\|Sy_n - x^*\| + \|y_n - x^*\|] + \|v_n\| \\
&\le (1 - \alpha_n)\beta_n[\|x_n - x^*\| + \|Sx_n - x^*\|] \\
&\qquad + (1 - \alpha_n)\|u_n\| + \alpha_n[2M + \|u_n\|] + \|v_n\| \\
&\le 2M(\alpha_n + \beta_n) + \|u_n\| + \|v_n\| \to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

and by the uniform continuity of T we obtain $\lim\|Sx_{n+1} - Sy_n\| = 0$. Set $\rho_n := \|x_n - x^*\|$, $\delta_n := \alpha_n k$, $\sigma_n := \alpha_n[5M\alpha_n + \|Sx_{n+1} - Sy_n\|]$, and $\lambda_n = 3\|v_n\|$ in (14) to obtain

$$\rho_{n+1} \le [1 - \delta_n]\rho_n + \sigma_n + \lambda_n \qquad \forall n \ge 0.$$

Clearly $\sum\delta_n = \infty$, $\sigma_n = o(\delta_n)$, $\sum\lambda_n < \infty$, and hence it follows from Lemma 2 of [18] that $\lim \rho_n = 0$, completing the proof of Theorem 3.

Let K be a nonempty subset of a Banach space E and $T\colon K \to E$ be a strong pseudocontraction. Then from (2) we obtain

$$\operatorname{Re}\langle (I - T)x - (I - T)y, j(x - y)\rangle \ge \frac{t - 1}{t}\|x - y\|^2.$$

Set $k = (t - 1)/t \in (0, 1)$, then

$$\operatorname{Re}\langle (I - T - kI)x - (I - T - kI)y, j(x - y)\rangle \ge 0,$$

and it follows from Lemma 1.1 of Kato that

$$\|x - y\| \le \|x - y + r[(I - T - kI)x - (I - T - kI)y]\|. \tag{15}$$

Using (2) and (15) the following corollaries follow easily as in the proofs of the previous theorems and the proofs are therefore omitted.

COROLLARY 2. *Suppose E is an arbitrary Banach space and K is a nonempty closed subset of E. Suppose* $T: K \to E$ *is a Lipschitz strongly pseudocontractive mapping and suppose T has a fixed point in K. Suppose* $\{u_n\}, \{v_n\}, \{\alpha_n\}$, *and* $\{\beta_n\}$ *are as in Theorem* 1, *and suppose the sequence* $\{x_n\}$ *generated from an arbitrary* $x_0 \in K$ *by*

$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n + u_n, \qquad n \geq 0$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n + v_n, \qquad n \geq 0$$

is contained in K. Then $\{x_n\}$ *converges strongly to the fixed point of T.*

COROLLARY 3. *Suppose E is a uniformly smooth Banach space and K is a nonempty closed subset of E. Suppose* $T: K \to E$ *is a strongly pseudocontractive mapping with a fixed point in K. Suppose* $\{u_n\}, \{v_n\}, \{\alpha_n\}, \{\beta_n\}$, *and* $\{x_n\}$ *are as in Corollary* 2, *and* $\{Tx_n\}$ *and* $\{Ty_n\}$ *are bounded. Then* $\{x_n\}$ *converges strongly to the fixed point of T.*

COROLLARY 4. *Suppose E and K are as in Corollary* 2 *and* $T: K \to E$ *is a uniformly continuous strongly pseudocontractive mapping with a fixed point in K. Suppose* $\{u_n\}, \{v_n\}, \{\alpha_n\}, \{\beta_n\}, \{x_n\}, \{Tx_n\}$, *and* $\{Ty_n\}$ *are as in Corollary* 3. *Then the sequence* $\{x_n\}$ *converges strongly to the fixed point of T.*

Remark 3. In [18] Liu stated a theorem (see Theorem 2 of [18]) similar to Corollary 2 in uniformly smooth Banach spaces but without the hypothesis that T has a fixed point in K. In his proof of the theorem he claimed that the existence and uniqueness of a fixed point of T are direct consequences of Proposition 3 of Martin [14]. This is false. If T has a fixed point, the uniqueness follows directly from the definition of T. However, the T in the theorem need not have a fixed point in K as can be seen from the following simple example:

EXAMPLE 1. Let $\Re$ denote the reals with the usual norm and $K = [0, 1]$. Define $T: [0, 1] \to \Re$ by $Tx = \frac{1}{2}x + 1$. Then T is a contraction and hence strongly pseudocontractive. Clearly T has no fixed point in K.

Even if T is a selfmapping of K, the following example shows that T still may fail to have a fixed point in K if K is not convex.

EXAMPLE 2. Let $K = \{1, 2\} \subseteq \Re$. Define $T: K \to K$ by $T(1) = 2$, $T(2) = 1$. Then T is strongly pseudocontractive and has no fixed point in K.

As a consequence of Remark 3, Corollary 2 extends Theorem 2 of Liu [18] from uniformly smooth Banach spaces to arbitrary Banach spaces. Moreover, the boundedness condition imposed on the range of T in Theorem 2 of [18] is not imposed in our Corollary 2. Also Theorem 4 of [18] is a special case of Corollary 3 of our result for which $\beta_n \equiv 0$.

COROLLARY 5. *Let E be an arbitrary Banach space and let $T: E \to E$ be a Lipschitz accretive operator. Let $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ be as in Theorem* 1. *Then for any given $f \in E$ the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in E$ by*

$$y_n = (1 - \beta_n)x_n + \beta_n(f - Tx_n) + u_n, \qquad n \geq 0$$
$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f - Ty_n) + v_n, \qquad n \geq 0$$

converges strongly to the solution of the equation $x + Tx = f$.

Proof. The existence of a solution follows from Martin [20] and the uniqueness follows from the accretivity of T. Let x^* denote the solution. Define $S: E \to E$ by $Sx = f - Tx$. Then x^* is a fixed point of S and S is Lipschitz with the same Lipschitz constant as T. Furthermore, for all $x, y \in E$ there exists $j(x - y) \in J(x - y)$ such that

$$\mathrm{Re}\langle (I - S)x - (I - S)y, j(x - y)\rangle \geq \|x - y\|^2.$$

The rest of the argument is now essentially the same as in the proof of Theorem 1 and is therefore omitted.

The proofs of the following corollaries—Corollaries 6 and 7—also follow as in the proofs of Theorems 2 and 3, respectively, and are therefore omitted.

COROLLARY 6. *Suppose E is a uniformly smooth Banach space and $T: E \to E$ is m-accretive. Suppose S, $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{x_n\}$ are as in Corollary* 5, *and that $\{Sx_n\}$ and $\{Sy_n\}$ are bounded. Then the sequence $\{x_n\}$ converges strongly to the solution of the equation $x + Tx = f$.*

COROLLARY 7. *Let E be an arbitrary Banach space and let $T: E \to E$ be a uniformly continuous accretive operator. Let S, $\{u_n\}$, $\{v_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{x_n\}$, $\{Sx_n\}$, and $\{Sy_n\}$ be as in Corollary* 6. *Then the sequence $\{x_n\}$ converges strongly to the solution of the equation $x + Tx = f$.*

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