Journal of Mathematical Analysis and Applications **236,** 48–73 (1999)
Article ID jmaa.1999.6421, available online at http://www.idealibrary.com on IDEAL®

A Generalized Steepest Descent Approximation for the Zeros of m-Accretive Operators

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Submitted by William F. Ames

Received December 17, 1998

A necessary and sufficient condition is proved for a generalized steepest descent approximation to converge to the zeros of m-accretive operators. Related results deal with the convergence of the scheme to fixed points of pseudocontractive maps.

Key Words: accretive operators; uniformly smooth space; uniformly convex space.

1. INTRODUCTION

Let X be a real normed linear space with dual X^*. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$Jx = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot , \cdot \rangle$ denotes the generalized duality pairing. It is well known that if X^* is strictly convex then J is single-valued and if X^* is uniformly convex then J is uniformly continuous on bounded subsets of X.

* Research supported by the fellowship from the OEA of the Abdus Salam ICTP, Trieste, Italy for the Ph. D. program in Mathematics for Sub-Saharan Africa, at the University of Nigeria, Nsukka, Nigeria.

0022-247X/99 $30.00

An operator A with domain $D(A)$ and range $R(A)$ in X is called *accretive* if, for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0. \tag{1}$$

Furthermore, A is called *strongly accretive* if, for each $x, y \in D(A)$, there exist $j(x-y) \in J(x-y)$ and a real number $k > 0$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq k\|x-y\|^2. \tag{2}$$

The operator A is said to be *ϕ-strongly accretive* if for each $x, y \in D(A)$ there exist $j(x-y) \in J(x-y)$ and a strictly increasing function ϕ: $[0, \infty) := \Re^+ \to \Re^-$ with $\phi(0) = 0$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq \phi(\|x-y\|)\|x-y\|. \tag{3}$$

The operator A is called *m-accretive* if it is accretive and $(I + rA)(D(A)) = X$ for all $r > 0$, where I denotes the identity operator on X. Let $N(A) := \{u \in D(A): Au = 0\} \neq \varnothing$. If the inequalities (1), (2), and (3) hold for all $x \in D(A)$ and $y \in N(A)$ then A is called *quasi-accretive*, *strongly quasi-accretive*, and *ϕ-strongly quasi-accretive*, respectively.

The accretive operators were introduced independently by Browder [2] and Kato [29] in 1967. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem

$$\begin{aligned} \frac{du}{dt} + Au &= 0 \\ u(0) &= u_0 \end{aligned} \tag{4}$$

is solvable if A is locally Lipschitzian and accretive. It is well known that many physically significant problems can be modeled in the form (4). Typical examples of how such evolution equations arise are found in models involving the heat, the wave, or the Schrödinger equation (see, e.g., [46]). If u is independent of t, then $Au = 0$ and the solution of this equation corresponds to the equilibrium points of the system (4). Consequently, considerable research efforts have been devoted to finding constructive methods for approximating solutions of the equation

$$Au = 0, \tag{5}$$

where A is an accretive-type operator on appropriate Banach spaces (see, e.g., [5, 7, 20, 22–27, 30, 32–39, 41, 42, 44, 45]).

Closely related to the class of accretive maps is the class of *pseudocontractive* operators. An operator T with domain $D(T)$ and range $R(T)$ in X

is called *strongly pseudocontractive* if, for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a constant $t > 1$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq t^{-1} \|x - y\|^2. \tag{6}$$

If $t = 1$ in Eq. (6), then T is called *pseudocontractive*. The map T is called *ϕ-strongly pseudocontractive* if for all $x, y \in D(T0$ there exist $j(x-y) \in J(x-y)$ and a strictly increasing function ϕ: $\Re^+ \to \Re^+$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|. \tag{7}$$

T is called *ϕ-hemicontractive* if the relation (7) holds for all $x \in D(T)$ and $y \in N(I - T)$. It follows from inequalities (3) and (7) that T is ϕ-strongly pseudocontractive if and only if $(I - T)$ is ϕ-strongly accretive, so that the mapping theory for accretive operators is intimately connected with the fixed point theory for pseudocontractions.

Two well-known iterative methods, the *Mann iteration method* (see, e.g., [31]) and the *Ishikawa iteration method* (see, e.g., [28, 40]), have successfully been employed for *self-maps* (see, e.g., [25, 27, 34, 35, 37, 45]). If $D(A)$ is a *proper* subset of X (and this is the case in several applications) both the Mann and Ishikawa iteration methods may not be well defined. Under this situation, for Hilbert spaces, this problem has been overcome by the introduction of the *proximity map* in the recursion formulas (see, e.g., [3, 6]). The advantage of this is that if K is a nonempty closed convex subset of a Hilbert space H and P_K: $H \to K$ is the proximity map of H onto K, then P_K is *nonexpansive* (i.e., $\|P_K x - P_K y\| \leq \|x - y\|$ for $x, y \in H$). This fact actually characterizes Hilbert spaces and consequently is not available in more general Banach spaces. In this connection, the following result which holds in certain Banach spaces is of interest.

LEMMA 1 (Reich [38]). *Let X be a Banach space which is both uniformly convex and uniformly smooth. Let A: $D(A) \subseteq X \to X$ be m-accretive and let $J_r = (I + rA)^{-1}$. Then for $x \in X$ the strong limit* $\lim_{r \to 0} J_r(x)$ *exists. Denote this strong limit by Qx. Then, Q: $X \to$* cl$(D(A))$ *is a nonexpansive retraction of X onto* cl$(D(A))$, *where* cl$(D(A))$ *represents the closure of the domain of A.*

It is known that *under the hypothesis of Lemma* 1, cl$(D(A))$ *is convex* (see, e.g., [1]).

In the rest of the paper Q will refer to the operator defined in this lemma.

In connection with the iterative approximation of the solution of Eq. (5), the following result for *self-maps* was recently proved.

THEOREM XR (Xu and Roach [45]). *Let X be a uniformly smooth Banach space and let A: $D(A) = X \to X$ be a quasi-accretive bounded operator such that if for any $x \in D(A)$, $p \in N(A)$, and any $j(x-p) \in J(x-p)$ the equality $\langle Ax, j(x-p)\rangle = 0$ holds if and only if $Ax = Ap = 0$, then, for any initial value $x_0 \in D(A)$, there is a positive real constant $T(x_0)$ such that the sequence $\{x_n\}$ generated from $\{x_0\}$ in $D(A)$ by $x_{n+1} = x_n - t_n Ax_n$, $n \geq 0$, where $t_n \in (0,\infty)$, $\Sigma t_n = \infty$, $t_n \to 0$ as $n \to \infty$, with $t_n \leq T(x_0)$ for any n, converges strongly to a solution x^* of the equation $Ax = 0$ if and only if there is a strictly increasing function ϕ: $\Re^+ \to \Re^+$, $\phi(0) = 0$, such that*

$$\langle Ax_n - Ax^*, j(x_n - x^*)\rangle \geq \phi(\|x_n - x^*\|)(\|x_n - x^*\|).$$

The main ingredient in the proof of Theorem XR is the following inequality which holds in real uniformly smooth Banach spaces X. For each $x, y \in X$,

$$\|x+y\| \leq \|x\|^2 + 2\langle y, j(x)\rangle + D\max\left\{\|x\| + \|y\|, \frac{c}{2}\right\}\rho_X(\|y\|),$$

where D and c are positive constants (see [44]). Other recent theorems related to Theorem XR can be found, for example, in [11, 25, 32, 14]. More recently, Morales and one of the authors [32] considered the following inequality due to Reich [39]. Let X be real uniformly smooth space. For each $x, y \in X$, the following inequality holds:

$$\|x+y\| \leq \|x\|^2 + 2\langle x, j(y)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|), \tag{8}$$

where b is a function which depends on the geometry of X (see, e.g., [39]).

They proved the following theorem.

THEOREM MC (Morales and Chidume [32]). *Let X be a uniformly smooth Banach space, let b be the function appearing in* (8), *and let A: $X \to X$ be a bounded demicontinuous mapping, which is also ϕ-strongly accretive on X. Let $z \in X$ and let x_0 be an arbitrary initial value in X for which the* $\liminf_{r\to\infty} \phi(r) > \|A(x_0)\|$. *Then the approximating scheme $x_{n+1} = x_n - c_n(Ax_n - z)$, $n = 0, 1, 2, \ldots$ converges strongly to the unique solution of the equation $Ax = z$, provided that the sequence $\{c_n\}$ of positive real numbers satisfies the following*: (i) *$\{c_n\}$ is bounded above by some constant r_0*; (ii) $\Sigma c_n = \infty$; *and* (iii) $\Sigma c_n b(c_n) < \infty$.

The condition $\Sigma c_n b(c_n) < \infty$ in Theorem MC is, in general, not convenient to verify in applications. Nevanlinna and Reich [33], however, have shown that, for any given continuous nondecreasing function b with $b(0) = 0$, sequences $\{\lambda_n\}$ always exist such that (i) $0 < \lambda_n < 1$, $n \geq 0$; (ii) $\Sigma\lambda_n = \infty$; (iii) $\Sigma\lambda_n b(\lambda_n) < \infty$. If $X = L_p(1 < p < \infty)$, we can choose any sequence $\{\lambda_n\}$ in $l^s \setminus l^1$, with $s = p$ if $1 < p \leq 2$ and $s = 2$ if $p \geq 2$.

In 1995, Liu [30] introduced what he called Ishikawa and Mann iteration processes "with errors" for nonlinear strongly accretive mappings as follows:

(a) For K a nonempty subset of a real Banach space X and $T: K \to X$, the sequence $\{x_n\}$ defined by $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n + u_n, \qquad y_n = (1 - \beta_n) x_n + \beta_n T x_n + v_n, \qquad n \geq 0,$$

where (i) $\{\alpha_n\}$ and $\{\beta_n\}$ are some real sequences in $(0,1)$ satisfying appropriate conditions, (ii) $\sum \|u_n\| < \infty$, $\sum \|v_n\| < \infty$, is called the *Ishikawa iteration process with errors*.

(b) With K, X, and T as in part (a) the sequence $\{x_n\}$ defined by $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n + u_n, \qquad n \geq 0,$$

where $\{\alpha_n\}$ and $\{u_n\}$ satisfy conditions as in part (a), is called the *Mann iteration process with errors*.

While it is well known that consideration of errors in iterative processes is an important aspect of the theory, it is also clear that the iteration process with errors introduced in (a) and (b) are unsatisfactory. The conditions $\sum \|u_n\| < \infty$, $\sum \|v_n\| < \infty$ imply, in particular, that the errors tend to zero. This is incompatible with the randomness of the occurrence of errors. Recently, Yuguang Xu [42] introduced the following satisfactory definitions.

(A) Let K be a nonempty convex subset of X and let $T: K \to K$ be a mapping. For any given $x_0 \in K$, the sequence $\{x_n\}$ defined iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \qquad y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \qquad n \geq 0, \tag{9}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $(0,1)$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in K for all integers $n \geq 0$, is called the *Ishikawa iteration sequence with errors*.

(B) If, with the same notation and definitions as in (A), $b'_n = c'_n = 0$ for all integers $n \geq 0$, then the sequence $\{x_n\}$ now defined by $x_0 \in K$, $x_{n+1} = a_n x_n + b_n T x_n + c_n u_n$, $n \geq 0$, is called the *Mann iteration sequence with errors*.

It is our purpose in this paper to construct an iterative process *with errors in the sense of* (A) *and* (B) which converges strongly to the solution of Eq. (5) where A is an accretive-type map defined on *proper* subsets of appropriate Banach spaces. Our theorems improve, generalize, and unify most of the results that have appeared for this large class of operators. In particular, Theorem XR, Theorem MC, Theorems 3 and 4 of [14], and the theorems of [11] and [25] are special cases of our theorems. Moreover, our method of proof, which is of independent interest, is much simpler than the methods used in [45], [25], or [11]. Furthermore, our theorems show also, in particular, that in Theorem MC the condition $\sum c_n b(c_n) < \infty$, which depends explicitly on the geometry of the underlying Banach space, is not needed.

2. PRELIMINARIES

In the rest of the paper we shall need the following preliminaries and lemma.

A Banach space X is called *smooth* if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $j \in X^*$ such that $\|j\| = \|j(x)\| = 1$ (see, e.g., [21]). The *modulus of smoothness* of X is the function $\rho_X\colon [0,\infty) \to [0,\infty)$ defined by

$$\rho_X^{(\tau)} = \sup\left\{\tfrac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = \tau\right\}.$$

A Banach space X is called *uniformly smooth* (see, e.g., [44]) if $\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0$, and, for $q > 1$, X is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho_X(\tau) \le c\tau^q, \qquad \tau \in [0,\infty).$$

It is well known (see, e.g., [43]) that

$$L_p \text{ (or } l_p) \text{ is } \begin{cases} p\text{-uniformly smooth if } 1 < p \le 2 \\ 2\text{-uniformly smooth if } p \ge 2. \end{cases}$$

The Banach space X is called *uniformly convex* if, given any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| \le 1$, $\|y\| \le 1$, and $\|x - y\| \ge \epsilon$ we have $\|\frac{1}{2}(x+y)\| \le 1 - \delta$. It is well known that L_p spaces $(1 < p < \infty)$ are uniformly convex. Consequently, L_p spaces $(1 < p < \infty)$ are both uniformly smooth and uniformly convex.

Condition (I). A accretive operator A will be said to justify Condition (I) if $N(A) \neq \emptyset$ and for any $Qy_n \in D(A)$, $p \in N(A)$, and any $j(Qy_n - p) \in J(Qy_n - p)$ the equality $\langle AQy_n, j(Qy_n - p)\rangle = 0$ holds if and only if $AQy_n = Ap = 0$, where $\{y_n\}$ is the sequence defined in (9).

Let X be a real Banach space. The *subdifferential* of a function f on X is a map $\partial f\colon X \to 2^{X^*}$ defined by

$$\partial f(x) = \{x^* \in X^* : f(y) \geq f(x) + \langle y - x, x' \rangle \text{ for all } y \in X\}.$$

It is well known that Jx is the subdifferential of the functional $\frac{1}{2}\|x\|^2$. An immediate consequence of this is the following lemma.

LEMMA 2. *Let X be a real Banach space. Then there exists $j(x+y) \in J(x+y)$ such that*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \qquad \textit{for all } x, y \in X.$$

We shall make use of this lemma in what follows.

3. MAIN RESULTS

Now, we prove the following theorems.

3.1. *Non-Self-Maps*

THEOREM 1. *Let X be a uniformly smooth and uniformly convex real Banach space. Let $A\colon D(A) \subseteq X \to X$ be a bounded m-accretive operator with closed domain $D(A)$ and let A satisfy Condition* (I). *Then there exists a constant $d_0 > 0$ such that for bounded sequences $\{u_n\}, \{v_n\}$ in $D(A)$ and real sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ satisfying the following conditions*:

(i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n,\ n \geq 0,$

(ii) $0 < b_n + c_n \leq d_0,\ 0 \leq b'_n + c'_n \leq d_0,\ n \geq 0,$

(iii) $\sum_{n=0}^{\infty} b_n = \infty,\ c'_n \leq c_n \leq \alpha_n^2$, *where* $\alpha_n = b_n + c_n,\ n \geq 0,$

(iv) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} c'_n = 0,$

the sequence $\{x_n\}$ generated from arbitrary x_0, v_0, u_0 in $D(A)$ by

$$\begin{aligned} x_{n+1} &= Qp_n, \\ p_n &= a_n x_n + b_n(I - A)y_n + c_n u_n, \\ y_n &= a'_n x_n + b'_n x_n + c'_n v_n, \qquad n \geq 0, \end{aligned} \tag{10}$$

converges strongly to the unique solution x^* of the equation $Au = 0$ if and only if there exists a strictly increasing and surjective function $\psi\colon [0,\infty) := \Re^+ \to \Re^-$ with $\psi(0) = 0$ such that

$$\langle Ay_n - Ax^*, j(y_n - x^*)\rangle \geq \psi(\|y_n - x^*\|)\|y_n - x^*\|. \tag{11}$$

Proof. Set $\alpha_n := b_n + c_n$ and $\beta_n := b'_n + c'_n$. Then (10) reduces to

$$\begin{aligned} x_{n+1} &= Qp_n \\ p_n &= x_n - \alpha_n Ay_n - U_n \\ y_n &= x_n - c'_n(x_n - v_n), \end{aligned} \tag{12}$$

where $U_n := c_n(y_n - Ay_n - u_n) + \alpha_n c'_n(x_n - v_n)$ and conditions (ii) and (iii) reduce to $0 < \alpha_n \le d_0$, $0 \le \beta_n \le d_0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$.

Necessity. Let $\{x_n\}$ converge strongly to the unique solution x^* of $Au = 0$. Observe that, since the domain $D(A)$ of A is convex, $y_n \in D(A)$ and $Qy_n = y_n$ for all $n \ge 0$. Since $\{v_n\}$ is bounded and $\lim c'_n = 0$, it follows that $y_n \to x^*$ as $n \to \infty$. Let $D := \sup\|y_n - x^*\| < \infty$. If $D = 0$, then $y_n = x^*$ for all $n \ge 0$ and (11) holds trivially. If $D > 0$, then for any $t \in (0, D)$, we define the set C_t as follows:

$$C_t := \{n \in N : \|y_n - x^*\| \ge t\},$$

where N is the set of all nonnegative integers. Since $y_n \to x^*$, for any $t \in (0, D)$, there exists a positive integer n_0 such that $\|y_n - x^*\| < t$ for all $n \ge n_0$. This implies that, for all $t \in (0, D)$, C_t is a finite subset of N and $C_{t_1} \subseteq C_{t_2}$ for all $t_1, t_2 \in (0, D)$, $t_1 \ge t_2$. Define

$$h(t) := \min_{n \in C_t} \left\{ \frac{\langle Ay_n - Ax^*, j(y_n - x^*)\rangle}{\|y_n - x^*\|}, t \in (0, D) \right\}. \tag{13}$$

Clearly, $h(t)$ is nonnegative and nondecreasing. We now prove that $h(t) > 0$ for any $t \in (0, D)$. Assume this is not the case. Then there exists $t_0 \in (0, D)$ such that $h(t_0) = 0$. Hence by (13) there exists $n_0 \in C_{t_0} \ne \emptyset$ such that $\langle Ay_{n_0}, j(y_{n_0} - x^*)\rangle / \|y_{n_0} - x^*\| = 0$ and $\|y_{n_0} - x^*\| \ge t_0 > 0$. Now since the operator A satisfies Condition (I), we therefore have $Ay_{n_0} = Ax^* = 0$. That is to say, y_{n_0} is a solution of the equation $Au = 0$. But this contradicts the fact that the solution is unique. Thus $h(t) > 0$ for any $t \in (0, D)$.

Define the function $\psi: [0, \infty) \to [0, \infty)$ as follows:

$$\psi(t) = \begin{cases} 0, & \text{if } t = 0, \\ \left(\dfrac{t}{1+t}\right) h(t), & \text{if } t \in (0, D), \\ \left(\dfrac{t}{1+D}\right) \sup\limits_{s<D} h(s), & \text{if } t \in [D, \infty). \end{cases}$$

Then, ψ is a strictly increasing and surjective function with $\psi(0) = 0$ and satisfies (11) (since $\psi(t) \le h(t)$ for all $t \in (0, D)$). This completes the proof of the "necessity."

Sufficiency. Case (i). If $Ay_n = 0$ for all $n \ge 0$, by the convexity of $D(A)$ it follows from (10) that the sequences $\{p_n\}$ and $\{y_n\}$ are in $D(A)$. Consequently, the map Q reduces to the identity map of $D(A)$ and (12) reduces to

$$x_{n+1} = x_n - c_n(y_n - u_n) - \alpha_n c'_n(x_n - v_n)$$
$$y_n = x_n - c'_n(x_n - v_n)$$

and using (11) we get that

$$\|y_n - x^*\| \le \psi^{-1}(\|Ay_n\|) = \psi^{-1}(0) = 0,$$

i.e., $y_n = x^*$ for all n. Moreover,

$$y_n = (1 - c'_n)x_n + c'_n v_n$$
$$\|x_n\| \le \frac{1}{(1 - c'_n)}\{\|x^*\| + c'_n\|v_n\|\}$$
$$\le \frac{1}{(1 - a^*)}\{\|x^*\| + a^* m\},$$

where $m := \max\{\sup\{\|v_n\|\}, \sup\{\|u_n\|\}\}$ and $a^* < 1$ is a constant (which exists by (iv)) such that, for some positive integer m_0, $c_n \le a^*$ for all $n \ge m_0$. So $\{x_n\}$ is bounded. Furthermore, using (12), we obtain (since $y_n = x^*$ for all n) for some constant $D \ge 0$ that

$$0 \le \|x_n - x^*\| \le \|y_n - x^*\| + c'_n\|x_n - v_n\|$$
$$\le c'_n(\|x_n\| + \|v_n\|) \le Dc'_n \to 0 \qquad \text{as } n \to \infty.$$

Hence $\{x_n\}$ converges to $x^* \in N(A)$. To show that x^* is unique, let $y^* \in N(A)$ be such that $y^* \ne x^*$. Then proceeding as in the above we get that $x_n \to y^*$, which contradicts the fact that $x_n \to x^*$. Thus the solution is unique.

Case (ii). Suppose there exists n such that $Ay_n \ne 0$. Without loss of generality we may assume that $Ay_0 \ne 0$. Then from (11) and (12) we get $\|y_0 - x^*\| \le \psi^{-1}(\|Ay_0\|)$ and

$$\|x_0 - x^*\| \le \|x_0 - y_0\| + \|y_0 - x^*\| \le c'_0(\|x_0\| + m) + \psi^{-1}(\|Ay_0\|)$$
$$\le \|x_0\| + m + \psi^{-1}(\|Ay_0\|).$$

Now set

$$\begin{aligned} a_0 &:= \|x_0\| + m + \psi^{-1}(\|Ay_0\|), \\ r(a_0) &:= \psi\left(\frac{a_0}{2}\right)\left(\frac{a_0}{8}\right) > 0, \\ M(x_0) &:= \sup\big\{\|Au\| : \|u - x_0\| \le 2\big(a_0 + r(a_0)\big)\big\}, \\ R(x_0) &:= 4\big\{M(x_0) + r(a_0) + a_0 + m + \|x_0\|\big\}. \end{aligned} \qquad (*)$$

Since j is uniformly continuous on bounded subsets of X, for $\epsilon := \psi(a_0/2)a_0/4M(x_0)$ there exists a $\delta > 0$ such that, for $x, y \in B(0, 2(a_0 + r(a_0)))$, $\|x - y\| < \delta$ implies $\|j(x) - j(y)\| < \epsilon$. Set

$$d_0 := \min\left\{\frac{a_0}{2R(x_0)}, \frac{r(a_0)}{R(x_0)(1 + a_0 + 2r(a_0))}, \frac{\delta}{3R(x_0)}\right\}. \qquad (14)$$

Claim 1. $\{x_n\}$ is bounded.

Suppose the sequence $\{x_n\}$ is not bounded. Let n_0 be the first natural number such that

$$\|x_{n_0} - x^*\| > a_0. \qquad \text{Then } \|x_{n_0 - 1} - x^*\| \le a_0. \qquad (15)$$

This implies that $\|x_{n_0-1} - x_0\| \le \|x_{n_0-1} - x^*\| + \|x^* - x_0\| \le 2a_0$, which gives that $\|Ax_{n_0-1}\| \le M(x_0)$ and $\|x_{n_0-1}\| \le 2a_0 + \|x_0\|$ and from (12), (14), and (15) we obtain that

$$\begin{aligned} \|y_{n_0-1} - x^*\| &\le \|x_{n_0-1} - x^*\| + c'_{n_0-1}\big(\|x_{n_0-1}\| + m\big) \\ &\le a_0 + d_0\big(2a_0 + \|x_0\| + m\big) \le a_0 + r(a_0) \\ \|y_{n_0-1} - x_0\| &\le \|y_{n_0-1} - x^*\| + \|x^* - x_0\| \le 2a_0 + r(a_0). \end{aligned}$$

Consequently, $\|Ay_{n_0-1}\| \le M(x_0)$ and $\|y_{n_0-1}\| \le 2a_0 + r(a_0) + \|x_0\|$.

Moreover, using (12) and condition (iii),

$$\begin{aligned} \|x_{n_0} - x^*\| \le \|x_{n_0-1} - x^*\| & \\ + \alpha_{n_0-1}\big\{\|Ay_{n_0-1}\| &+ \|y_{n_0-1}\| + \|Ay_{n_0-1}\| + \|u_{n_0-1}\| \\ &+ \|x_{n_0-1}\| + \|v_{n_0-1}\|\big\} \\ \le a_0 + \alpha_{n_0-1}&\big\{2M(x_0) + 4a_0 + r(a_0) + 2\|x_0\| + 2m\big\} \\ \le a_0 + d_0 R(x_0) &\le a_0 + r(a_0) \end{aligned}$$

and hence

$$\|x_{n_0} - x_0\| \le \|x_{n_0} - x^*\| + \|x^* - x_0\| \le 2a_0 + r(a_0),$$

which implies $\|Ax_{n_0}\| \le M(x_0)$ and $\|x_{n_0}\| \le 2a_0 + r(a_0) + \|x_0\|$.

Furthermore,

$$\begin{aligned} \|y_{n_0} - x^*\| &\le \|x_{n_0} - x^*\| + c'_{n_0}\big(\|x_{n_0}\| + m\big) \\ &\le a_0 + r(a_0) + d_0 R(x_0) \le a_0 + 2r(a_0) \\ \|y_{n_0} - x_0\| &\le \|y_{n_0} - x^*\| + \|x^* - x_0\| \le 2\big(a_0 + r(a_0)\big). \end{aligned}$$

Consequently $\|Ay_{n_0}\| \le M(x_0)$ and $\|y_{n_0}\| \le 2(a_0 + r(a_0)) + \|x_0\|$.

Now, from (12), (11), Lemma 2, and the above relations we obtain that

$$\begin{aligned} \|x_{n_0+1} - x^*\|^2 &\le \|p_{n_0} - x^*\|^2 = \|x_{n_0} - x^* - \alpha_{n_0} Ay_{n_0} - U_{n_0}\|^2 \\ &\le \|x_{n_0} - x^*\|^2 \\ &\quad - 2\alpha_{n_0}\big\langle Ay_{n_0} - Ax^*, j(p_{n_0} - x^*) - j(y_{n_0} - x^*)\big\rangle \\ &\quad - 2\alpha_{n_0}\big\langle Ay_{n_0} - Ax^*, j(y_{n_0} - x^*)\big\rangle \\ &\quad - 2\big\langle U_{n_0}, j(p_{n_0} - x^*)\big\rangle \\ &\le \|x_{n_0} - x^*\|^2 + 2\alpha_{n_0}\|Ay_{n_0}\|\,\big\|j(p_{n_0} - x^*) - j(y_{n_0} - x^*)\big\| \\ &\quad - 2\alpha_{n_0}\psi\big(\|y_{n_0} - x^*\|\big)\|y_{n_0} - x^*\| + 2\|U_{n_0}\|\,\|p_{n_0} - x^*\|. \end{aligned} \tag{16}$$

Observe that from (14), condition (ii), and (12)

$$\begin{aligned} \|y_{n_0} - x^*\| &\ge \|x_{n_0} - x^*\| - \|y_{n_0} - x_{n_0}\| \\ &\ge a_0 - \Big(c'_{n_0}\big(\|x_{n_0}\| + \|v_{n_0}\|\big)\Big) \ge a_0 - \left(\frac{a_0}{2}\right) = \left(\frac{a_0}{2}\right) \\ \|U_{n_0}\| &\le c_{n_0}\big(\|y_{n_0}\| + \|Ay_{n_0}\| + m\big) + \alpha_{n_0} c'_{n_0}\big(\|x_{n_0}\| + m\big) \\ &\le c_{n_0}\big\{4a_0 + 3r(a_0) + M(x_0) + 2\|x_0\| + 2m\big\} \\ &\le \alpha_{n_0}^2 R(x_0) \qquad \big(\text{since } c'_n \le c_n \le \alpha_n^2\big) \\ \|p_{n_0} - x^*\| &\le \|x_{n_0} - x^*\| + \alpha_{n_0}\|Ay_{n_0}\| + \|U_{n_0}\| \\ &\le a_0 + r(a_0) + \alpha_{n_0}\big(M(x_0) + r(a_0)\big) \le a_0 + 2r(a_0). \end{aligned}$$

Thus (16), (14), and the above estimates give that

$$\begin{aligned}\|x_{n_0+1} - x^*\|^2 &\le \|x_{n_0} - x^*\|^2 + 2\alpha_{n_0} M(x_0) \| j(p_{n_0} - x^*) - j(y_{n_0} - x^*)\| \\ &\quad + 2\alpha_{n_0}^2 R(x_0)[a_0 + 2r(a_0)] - 2\alpha_{n_0}\psi\left(\frac{a_0}{2}\right)\left(\frac{a_0}{2}\right) \\ &\le \|x_{n_0} - x^*\|^2 + 2\alpha_{n_0} M(x_0) \| j(p_{n_0} - x^*) - j(y_{n_0} - x^*)\| \\ &\quad + 2\alpha_{n_0} r(a_0) - \alpha_{n_0}\psi\left(\frac{a_0}{2}\right) a_0. \qquad (17)\end{aligned}$$

Using condition (ii) and noting that

$$\begin{aligned}\|p_{n_0} - y_{n_0}\| &\le \|x_{n_0} - y_{n_0}\| + \alpha_{n_0}\|Ay_{n_0}\| + \|U_{n_0}\| \\ &\le c'_{n_0}\big(\|x_{n_0}\| + m\big) + \alpha_{n_0}\big(M(x_0) + r(a_0)\big) \le \delta/3 + \delta/3 < \delta\end{aligned}$$

and that $(p_{n_0} - x^*), (y_{n_0} - x^*) \in B(0, 2(a_0 + r(a_0))$, we get by the uniform continuity of j on bounded subsets of X that

$$\| j(p_{n_0} - x^*) - j(y_{n_0} - x^*)\| \le \frac{\psi(a_0/2)a_0}{4M(x_0)}.$$

Substituting this in (17) we get, using the definition of $r(a_0)$ from $(*)$, that

$$\begin{aligned}\|x_{n_0+1} - x^*\|^2 &\le \|x_{n_0} - x^*\|^2 + \alpha_{n_0}\psi\left(\frac{a_0}{2}\right)\frac{a_0}{2} + 2\alpha_{n_0} r(a_0) - \alpha_{n_0}\psi\left(\frac{a_0}{2}\right)a_0 \\ &\le \|x_{n_0} - x^*\|^2 - \frac{1}{4}\alpha_{n_0}\psi\left(\frac{a_0}{2}\right)a_0 \le \|x_{n_0} - x^*\|^2\end{aligned}$$

and hence $\|x_{n_0+1} - x^*\| \le \|x_{n_0} - x^*\|$. Consequently,

$$\|x_{n_0+1} - x_0\| \le \|x_{n_0+1} - x^*\| + \|x_0 - x^*\| \le 2a_0 + r(a_0)$$

and

$$\|Ax_{n_0+1}\| \le M(x_0).$$

To complete the proof of Claim 1, let $\rho_n := \|x_n - x^*\|$. If we assume that $\rho_{n_0+1} > a_0$, then, by the previous argument, we get $\rho_{n_0+2} \le \rho_{n_0+1}$. On the other hand, if $\rho_{n_0+1} \le a_0$, then either $\rho_n \le a_0$ for all $n \ge n_0 + 1$, in which case the proof is complete, or there exists a positive integer j such that $\rho_j > a_0$ while $\rho_{j-1} \le a_0$. In the latter case, if $\|Ax_{j-1}\| \le M(x_0)$ and $\|Ay_{j-1}\| \le M(x_0)$ we return to the previous argument. To this end, note that $\rho_{j-1} \le a_0$ implies that $\|x_{j-1} - x_0\| \le \|x_{j-1} - x^*\| + \|x_0 - x^*\| \le 2a_0$

and so $\|Ax_{j-1}\| \le M(x_0)$. Moreover,

$$\begin{aligned}\|y_{j-1} - x^*\| &= \|(x_{j-1} - x^*) + c'_{j-1}(x_{j-1} - v_{j-1})\| \\ &\le \|x_{j-1} - x^*\| + c'_{j-1}(\|x_{j-1}\| + \|v_{j-1}\|) \\ &\le a_0 + r(a_0) \qquad \text{since } c'_{j-1} \le d_0.\end{aligned}$$

Hence $\|y_{j-1} - x_0\| \le \|y_{j-1} - x^*\| + \|x_0 - x^*\| \le 2a_0 + r(a_0)$, so that $\|Ay_{j-1}\| \le M(x_0)$. Thus, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{p_n\}$, and $\{Ay_n\}$ are bounded. This completes the proof of Claim 1.

Now, observe that, by condition (iii) ($c'_n \le c_n \le \alpha_n^2$), and using the boundedness established above there exists a constant $M_1 \ge 0$ such that

$$\begin{aligned}\|U_n\| \cdot \|p_n - x^*\| &\le \{c_n(\|y_n\| + \|Ay_n\| + m) + \alpha_n c'_n(\|x_n\| + m)\}\|p_n - x^*\| \\ &\le c_n M_1 \le \alpha_n^2 M_1.\end{aligned}$$

By (11), (12), and Lemma 2 together with the above estimates, we get that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|p_n - x^*\|^2 = \|(x_n - x^*) - \alpha_n(Ay_n - Ax^*) - U_n\|^2 \\ &\le \|x_n - x^*\|^2 \\ &\quad - 2\alpha_n\langle Ay_n - Ax^*, j(p_n - x^*) - j(y_n - x^*)\rangle \\ &\quad - 2\alpha_n\langle Ay_n - Ax^*, j(y_n - x^*)\rangle - 2\langle U_n, j(p_n - x^*)\rangle \\ &\le \|x_n - x^*\|^2 + 2\alpha_n\|Ay_n\|\,\|j(p_n - x^*) - j(y_n - x^*)\| \\ &\quad - 2\alpha_n\psi(\|y_n - x^*\|)\|y_n - x^*\| + 2\|U_n\|\,\|p_n - x^*\| \\ &\le \|x_n - x^*\|^2 \\ &\quad + 2\alpha_n(\|Ay_n\|\,\|j(p_n - x^*) - j(y_n - x^*)\| + \alpha_n M_1) \\ &\quad - 2\alpha_n\psi(\|y_n - x^*\|)\|y_n - x^*\|. \end{aligned} \tag{18}$$

Moreover, since, from (12),

$$\begin{aligned}\liminf_{n\to\infty}\|y_n - x^*\| &= \liminf_{n\to\infty}(\|y_n - x^*\| - c'_n\|x_n - v_n\|) \\ &\le \liminf_{n\to\infty}\|x_n - x^*\| = \liminf_{n\to\infty}(\|x_n - x^*\| - c'_n\|x_n - v_n\|) \\ &\le \liminf_{n\to\infty}\|y_n - x^*\|,\end{aligned}$$

we have

$$\liminf_{n\to\infty}\|y_n - x^*\| = \liminf_{n\to\infty}\|x_n - x^*\|.$$

Let $\liminf_{n\to\infty}\|y_n - x^*\| = \delta$ (say) ≥ 0.

Claim 2. $\delta = 0$.

Suppose not. Then there exists an integer $N_1 > 0$ such that

$$\psi(\|y_n - x^*\|)\|y_n - x^*\| \geq \frac{\delta}{2}\psi\left(\frac{\delta}{2}\right)$$

for all $n \geq N_1$. Since $\{Ay_n\}, \{p_n - x^*\}, \{y_n - x^*\}$ are bounded and $\|(p_n - x^*) - (y_n - x^*)\| \to 0$ as $n \to \infty$, by the uniform continuity of j on bounded subsets of X, there exists a positive integer N_2 such that

$$\|Ay_n\|\,\|j(p_n - x^*) - j(y_n - x^*)\| \leq \frac{\delta}{8}\psi\left(\frac{\delta}{2}\right)$$

for $n \geq N_2$ and also, by (iv), there exists a positive integer $N_3 > 0$ such that

$$\alpha_n M_1 \leq \frac{\delta}{8}\psi\left(\frac{\delta}{2}\right) \qquad \text{for all } n \geq N_3.$$

So, for all $n \geq N := \max\{N_1, N_2, N_3\}$, inequality (18) implies that

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \alpha_n\frac{\delta}{2}\psi\left(\frac{\delta}{2}\right).$$

This implies that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Hence we have that, for $N := \max\{N_1, N_2, N_3\}$,

$$\frac{\delta}{2}\psi\left(\frac{\delta}{2}\right)\sum_{n=N}^{\infty}\alpha_n \leq \|x_N - x^*\|^2,$$

which contradicts condition (iii) that $\sum_{n=0}^{\infty}\alpha_n = \infty$. This contradiction implies that $\delta = 0$.

Claim 3. $\{x_n\}$ converges to $x^* \in N(A)$.

Since $\liminf_{n\to\infty}\|x_n - x^*\| = 0$, there exists a subsequence $\{\|x_{n_j} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ such that $\lim_{j\to\infty}\|x_{n_j} - x^*\| = 0$. It follows that, given $\epsilon > 0$, there exists a positive integer j_0 such that $\|x_{n_j} - x^*\| < \epsilon$ for all $j \geq j_0$ $(n_j \geq n_{j_0})$. Set $\lambda_n := 2(\|Ay_n\|\,\|j(p_n - x^*) - j(y_n - x^*)\| + \alpha_n M_1)$ and observe that $\lambda_n \to 0$ as $n \to \infty$. Then there exists a positive integer N_4

such that $\lambda_n \le \frac{\epsilon}{4}\psi(\frac{\epsilon}{2})$, $\alpha_n \le \epsilon/8(M_2 + M_3 + M_4)$, for all $n \ge N_4$ where $M_2 := \sup\{\|Ay_n\| : n \ge 0\}$, $M_3 := \sup\{\|x_n - v_n\| : n \ge 0\}$ and M_4 is a constant such that

$$\|U_n\| \le c_n(\|y_n\| + \|Ay_n\| + m) + \alpha_n c'_n(\|x_n\| + m) \le c_n M_4 \le \alpha_n^2 M_4.$$

As $\lim_{j\to\infty} n_j = \infty$, we can choose j_* such that $N_{j_*} \ge \max\{n_{j_0}, N_4\}$ so that

$$\|x_{n_{j_*}} - x^*\| < \epsilon, \quad \text{and}$$

$$\lambda_n \le \frac{\epsilon}{4}\psi\left(\frac{\epsilon}{2}\right), \qquad \alpha_n \le \frac{\epsilon}{8(M_2 + M_3 + M_4)}, \qquad \text{for all } n \ge N_{j_*}.$$

We prove that $\|x_{n_{j_*+p}} - x^*\| < \epsilon$ for all positive integers $p \ge 1$. We proceed by induction on p. For $p = 1$, we prove that $\|x_{n_{j_*+1}} - x^*\| < \epsilon$. Suppose

$$\|x_{n_{j_*+1}} - x^*\| \ge \epsilon. \tag{19}$$

Then, using (12) we get that

$$\begin{aligned}\|x_{n_{j_*}} - x^*\| &\ge \|x_{n_{j_*+1}} - x^*\| - \alpha_{n_{j_*}}\|Ay_{n_{j_*}}\| - \|U_{n_{j_*}}\| \\ &\ge \epsilon - \alpha_{n_{j_*}} M_2 - \alpha_{n_{j_*}}^2 M_4 \ge \epsilon - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{3}{4}\epsilon\end{aligned}$$

and

$$\|y_{n_{j_*}} - x^*\| \ge \|x_{n_{j_*}} - x^*\| - c'_{n_{j_*}}\|x_{n_{j_*}} - v_{n_{j_*}}\| \ge \frac{3}{4}\epsilon - \alpha_{n_{j_*}}^2 M_3 > \frac{\epsilon}{2}.$$

Since ψ is strictly increasing, we have that $\psi(\|y_{n_{j_*}} - x^*\|) \ge \psi(\frac{\epsilon}{2})$. Inequalities (18) and (19) give that

$$\epsilon^2 \le \|x_{n_{j_*+1}} - x^*\|^2 < \epsilon^2 + \alpha_{n_{j_*}}\frac{\epsilon}{4}\psi\left(\frac{\epsilon}{2}\right) - \alpha_{n_{j_*}}\epsilon\psi\left(\frac{\epsilon}{2}\right) \le \epsilon^2,$$

a contradiction, so that $\|x_{n_{j_*+1}} - x^*\| < \epsilon$. Now, assume that, for some $p_0 > 1$, $\|x_{n_{j_*+p_0}} - x^*\| < \epsilon$. We prove that $\|x_{n_{j_*+p_0+1}} - x^*\| < \epsilon$. Assume for contradiction that $\|x_{n_{j_*+p_0+1}} - x^*\| \ge \epsilon$. Then from (12) we obtain

$$\begin{aligned}\|x_{n_{j_*+p_0}} - x^*\| &\ge \|x_{n_{j_*+p_0+1}} - x^*\| - \alpha_{n_{j_*+p_0}}\|Ay_{n_{j_*+p_0}}\| - \|U_{n_{j_*+p_0}}\| \\ &\ge \epsilon - \alpha_{n_{j_*+p_0}} M_2 - \alpha_{n_{j_*+p_0}}^2 M_4 \ge \tfrac{3}{4}\epsilon,\end{aligned}$$

and

$$\|y_{n_{j_*+p_0}} - x^*\| \geq \|x_{n_{j_*+p_0}} - x^*\| - c'_{n_{j_*+p_0}}\|x_{n_{j_*+p_0}} - v_{n_{j_*+p_0}}\|$$
$$\geq \frac{3}{4}\epsilon - \alpha^2_{n_{j_*+p_0}} M_3 > \frac{\epsilon}{2}.$$

Hence $\psi(\|y_{n_{j_*+p_0}} - x^*\|) \geq \psi(\frac{\epsilon}{2})$, and so inequality (18) gives

$$\epsilon^2 \leq \|x_{n_{j_*+p_0+1}} - x^*\| < \epsilon^2 + \alpha_{n_{j_*+p_0}} \frac{\epsilon}{4}\psi\left(\frac{\epsilon}{2}\right) - \alpha_{n_{j_*+p_0}} \epsilon\psi\left(\frac{\epsilon}{2}\right) \leq \epsilon^2,$$

a contradiction, and so $\|x_{n_{j_*+p_0+1}} - x^*\| < \epsilon$. Hence, $\|x_{n_{j_*+p}} - x^*\| < \epsilon$ for all positive integers $p \geq 1$, and this implies that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Uniqueness follows as in Case (i). This completes the proof of the sufficiency, completing the proof of Theorem 1. ∎

THEOREM 2. *Let X be a real uniformly smooth and uniformly convex Banach space. Let A: $D(A) \subseteq X \to X$ be a bounded m-accretive operator with closed domain $D(A)$. Let $N(A) \neq \phi$. Then there exists a constant $d_0 > 0$ such that for bounded sequences $\{u_n\}, \{v_n\}$ in $D(A)$ and real sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ satisfying the following conditions*:

(i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$, $n \geq 0$,

(ii) $0 < b_n + c_n \leq d_0$, $0 \leq b'_n + c'_n \leq d_0$, $n \geq 0$,

(iii) $\sum_{n=0}^{\infty} b_n + \infty$, $c'_n \leq c_n \leq \alpha_n^2$, *where* $\alpha_n := b_n + c_n$, $n \geq 0$,

(iv) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} c'_n = 0$,

the iteration process defined for any initial guesses x_0, v_0, u_0 in $D(A)$ by

$$\begin{aligned} x_{n=1} &= Qp_n, \\ p_n &= a_n x_n + b_n(I - A)Qy_n + c_n u_n, \\ y_n &= a'_n x_n + b'_n(I - A)x_n + c'_n v_n, \qquad n \geq 0, \end{aligned} \tag{20}$$

converges strongly to the unique solution x^* of the equation $Au = 0$ if there exists a strictly increasing and surjective function ψ: $\Re^+ \to \Re^+$ with $\psi(0) = 0$ such that

$$\begin{aligned} \left\langle Ax_{m_0} - Ax^*, j(x_{m_0} - x^*)\right\rangle &\geq \psi\left(\|x_{m_0} - x^*\|\right)\|x_{m_0} - x^*\|, \\ \left\langle AQy_n - Ax^*, j(Qy_n - x^*)\right\rangle &\geq \psi\left(\|Qy_n - x^*\|\right)\|Qy_n - x^*\|, \end{aligned} \tag{21}$$

where m_0 is the smallest positive integer such that $Ax_{m_0} \neq 0$.

Proof. The proof follows closely the proof of Theorem 1, so we shall omit some of the details. By letting $\alpha_n := b_n + c_n$ and $\beta_n := b'_n + c'_n$, (20) reduces to

$$\begin{aligned} x_{n=1} &= Qp_n, \\ p_n &= x_n - \alpha_n AQy_n - \alpha_n(x_n - Qy_n) - U_n, \\ y_n &= x_n - \beta_n Ax_n - c'_n(x_n - Ax_n - v_n), \qquad n \geq 0, \end{aligned} \tag{22}$$

where $U_n = c_n(Qy_n - AQy_n - u_n)$.

If $Ax_n = 0$ for all $n \geq 0$, then (20) reduces to (10) and the conclusion follows from Theorem 1. Suppose there exists n such that $Ax_n \neq 0$. Then without loss of generality we may assume that $Ax_0 \neq 0$. From (21) we get $\|x_0 - x^*\| \leq \psi^{-1}(\|Ax_0\|)$.

Now set

$$\begin{aligned} m &:= \max\{\sup\{\|u_n\|\}, \sup\{\|v_n\|\}\}. \\ a_0 &:= \|x_{n_0}\| + m + \psi^{-1}(\|Ax_0\|); \\ r(a_0) &:= \psi\left(\frac{a_0}{2}\right)\left(\frac{a_0}{8}\right) > 0; \\ M(x_0) &:= \sup\{\|Au\| : \|u - x_0\| \leq 2(a_0 + r(a_0))\}; \\ R(x_0) &:= 4\{M(x_0) + r(a_0) + a_0 + m + \|x_0\|\}. \end{aligned}$$

By the uniform continuity of j on bounded subsets of X, for $\epsilon := \psi(\frac{1}{2}a_0)a_0/4M(x_0)$ there exists a $\delta > 0$ such that, for $x, y \in B(0, 2(a_0 + r(a_0))$, $\|x - y\| < \delta$ implies $\|j(x) - j(y)\| < \epsilon$. Set

$$d_0 := \min\left\{\frac{a_0}{2R(x_0)}, \frac{r(a_0)}{R(x_0)(1 + a_0 + 2r(a_0))}, \frac{\delta}{3R(x_0)}\right\}.$$

Claim. $\{x_n\}$ is bounded.

Suppose the sequence $\{x_n\}$ is not bounded. Let n_0 be the first natural number for which

$$\|x_{n_0} - x^*\| > a_0. \qquad \text{Then } \|x_{n_0-1} - x^*\| \leq a_0. \tag{23}$$

Moreover, $\|x_{n_0-1} - x_0\| \leq \|x_{n_0-1} - x^*\| + \|x^* - x_0\| \leq 2a_0$, which gives that $\|Ax_{n_0-1}\| \leq M(x_0)$ and $\|x_{n_0-1}\| \leq 2a_0 + \|x_0\|$ and by (22), (23), and the above estimates we get that

$$\begin{aligned}\|Qy_{n_0-1}-x^*\| &\le \|y_{n_0-1}-x^*\| \\ &= \Big\| x_{n_0-1}-x^*-\beta_{n_0-1}Ax_{n_0-1} \\ &\quad -c'_{n_0-1}\big(x_{n_0-1}-Ax_{n_0-1}-v_{n_0-1}\big)\Big\| \\ &\le \|x_{n_0-1}-x^*\|+\beta_{n_0-1}\big\{2M(x_0)+\|x_{n_0-1}\|+m\big\} \\ &\le a_0+r(a_0)\end{aligned}$$

$$\|Qy_{n_0-1}-x_0\| \le \|y_{n_0-1}-x_0\| \le \|y_{n_0-1}-x^*\|+\|x^*-x_0\| \le 2a_0+r(a_0).$$

Consequently, $\|AQy_{n_0-1}\| \le M(x_0)$ and $\|y_{n_0-1}\| \le 2a_0+r(a_0)+\|x_0\|$.

Again by (22) and the above estimates we obtain, as in the proof of Theorem 1,

$$\begin{aligned}\|x_{n_0}-x^*\| \le \|x_{n_0-1}-x^*\| &+ \alpha_{n_0-1}\big(\|AQy_{n_0-1}\| \\ &+\|x_{n_0-1}\|+\|Qy_{n_0-1}\|\big)+\|U_{n_0-1}\| \le a_0+2r(a_0)\end{aligned}$$

and hence $\|x_{n_0}-x_0\| \le \|x_{n_0}-x^*\|+\|x^*-x_0\| \le 2a_0+2r(a_0)$, which implies $\|Ax_{n_0}\| \le M(x_0)$ and $\|x_{n_0}\| \le 2a_0+2r(a_0)+\|x_0\|$.

Moreover,

$$\begin{aligned}\|Qy_{n_0}-x^*\| &\le \|y_{n_0}-x^*\| \le \|x_{n_0}-x^*\|+\beta_{n_0}\big(2\|Ax_{n_0}\|+\|x_{n_0}\|+m\big) \\ &\le a_0+2r(a_0)+\beta_{n_0}\big\{2M(x_0)+2a_0+2r(a_0)+\|x_0\|+m\big\} \\ &\le a_0+2r(a_0)\end{aligned}$$

$$\|Qy_{n_0}-x_0\| \le \|y_{n_0}-x_0\| \le \|y_{n_0}-x^*\|+\|x^*-x_0\| \le 2\big(a_0+r(a_0)\big),$$

and hence $\|AQy_{n_0}\| \le M(x_0)$ and $\|y_{n_0}\| \le 2(a_0+r(a_0))+\|x_0\|$.

Now, by (22), (21), Lemma 2, and the above relations, and proceeding as in the proof of Theorem 1 we obtain

$$\begin{aligned}\|x_{n_0+1}-x^*\|^2 \le \|x_{n_0}-x^*\|^2 & \\ &+2\alpha_{n_0}\|AQy_{n_0}\|\,\big\|j(p_{n_0}-x^*)-j(Qy_{n_0}-x^*)\big\| \\ &-2\alpha_{n_0}\psi\big(\|Qy_{n_0}-x^*\|\big)\|Qy_{n_0}-x^*\| \\ &+2\alpha_{n_0}\|x_{n_0}-Qy_{n_0}\|\,\|p_{n_0}-x^*\|+2\|U_{n_0}\|\,\|p_{n_0}-x^*\|.\end{aligned} \tag{24}$$

Moreover, we have the following estimates

$$\begin{aligned}\|Qy_{n_0} - x^*\| &\geq \|x_{n_0} - x^*\| - \|Qy_{n_0} - x_{n_0}\| \geq \|x_{n_0} - x^*\| - \|y_{n_0} - x_{n_0}\| \\ &\geq a_0 - \beta_{n_0}\big(2\|Ax_{n_0}\| + \|x_{n_0}\| + \|v_{n_0}\|\big) \geq a_0 - \left(\frac{a_0}{2}\right) = \left(\frac{a_0}{2}\right) \\ \|U_{n_0}\| &\leq c_{n_0}\big(\|Qy_{n_0}\| + \|AQy_{n_0}\| + m\big) \\ &\leq c_{n_0}R(x_0) \leq \alpha_{n_0}^2 R(x_0) \leq \alpha_{n_0} r(a_0). \\ \|p_{n_0} - x^*\| &\leq \|x_{n_0} - x^*\| + \alpha_{n_0}\big(\|AQy_{n_0}\| + \|x_{n_0} - Qy_{n_0}\|\big) + \|U_{n_0}\| \\ &\leq a_0 + 2r(a_0) + \alpha_{n_0}\{M(x_0) + 2r(a_0)\} \leq a_0 + 2r(a_0).\end{aligned}$$

Moreover,

$$\begin{aligned}&2\alpha_{n_0}\|x_{n_0} - Qy_{n_0}\| \cdot \|p_{n_0} - x^*\| + 2\|U_{n_0}\| \cdot \|p_{n_0} - x^*\| \\ &\quad\leq 2\alpha_{n_0}\big\{\|x_{n_0} - Qy_{n_0}\| + \alpha_{n_0}R(x_0)\big\}\|p_{n_0} - x^*\| \\ &\quad\leq 2\alpha_{n_0}\big\{\|x_{n_0} - y_{n_0}\| + \alpha_{n_0}R(x_0)\big\}\|p_{n_0} - x^*\| \\ &\quad\leq 4\alpha_{n_0} r(a_0) \\ &\quad= \alpha_{n_0}\psi\left(\frac{a_0}{2}\right)\left(\frac{a_0}{2}\right).\end{aligned}$$

Thus (24), (21), and the above estimates give that

$$\begin{aligned}\|x_{n_0+1} - x^*\|^2 \leq \|x_{n_0} - x^*\|^2 &+ 2\alpha_{n_0}M(x_0)\big\|j(p_{n_0} - x^*) - j\big(Qy_{n_0} - x^*\big)\big\| \\ &- \alpha_{n_0}\psi\left(\frac{a_0}{2}\right)\left(\frac{a_0}{2}\right).\end{aligned} \tag{25}$$

Noting that by (22) and condition (ii)

$$\|p_{n_0} - Qy_{n_0}\| < \delta$$

and that $(p_{n_0} - x^*), (Qy_{n_0} - x^*) \in B(0, 2(a_0 + r(a_0))$, we get by the uniform continuity of j that

$$\big\|j(p_{n_0} - x^*) - j\big(Qy_{n_0} - x^*\big)\big\| \leq \frac{\psi(a_0/2)a_0}{4M(x_0)}.$$

Substituting this in (25) we get

$$\|x_{n_0+1} - x^*\|^2 \le \|x_{n_0} - x^*\|^2$$

and hence $\|x_{n_0+1} - x^*\| \le \|x_{n_0} - x^*\|$. Consequently,

$$\|x_{n_0+1} - x_0\| \le \|x_{n_0+1} - x^*\| + \|x_0 - x^*\| \le 2\{a_0 + r(a_0)\}$$

and

$$\|Ax_{n_0+1}\| \le M(x_0).$$

The rest of the argument follows as in the proof of Theorem 1 to yield that $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\}$, $\{Qy_n\}$, $\{p_n\}$, $\{Ax_n\}$, and $\{AQy_n\}$ are bounded.

Again by (22), (21), and Lemma 2 together with the above estimates, we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 & \\ &+ 2\alpha_n\{\|AQy_n\|\,\|j(p_n - x^*) - j(Qy_n - x^*)\| + \alpha_n M_1 \\ &\qquad + \|x_n - Qy_n\|\,\|p_n - x^*\|\} \\ &- 2\alpha_n \psi(\|Qy_n - x^*\|)\|Qy_n - x^*\|, \end{aligned} \tag{26}$$

where $M_1 > 0$ such that

$$\|U_n\|\,\|p_n - x^*\| \le c_n M_1 \le \alpha_n^2 M_1.$$

Observe that

$$\liminf_{n\to\infty}\|Qy_n - x^*\| = \liminf_{n\to\infty}\|y_n - x^*\| = \liminf_{n\to\infty}\|x_n - x^*\|.$$

Let $\liminf_{n\to\infty}\|Qy_n - x^*\| = \delta \ge 0$.

As in the proof of Theorem 1, $\delta = 0$. Since $\liminf_{n\to\infty}\|x_n - x^*\| = 0$, there exists a subsequence $\{\|x_{n_j} - x^*\|\}$ of the sequence $\{\|x_n - x^*\|\}$ such that $\lim_{j\to\infty}\|x_{n_j} - x^*\| = 0$. It follows that, given $\epsilon > 0$, there exists a positive integer j_0 such that $\|x_{n_j} - x^*\| < \epsilon$ for all $j \ge j_0$ $(n_j \ge n_{j_0})$. Set $\lambda_n := 2(\|AQy_n\|\,\|j(p_n - x^*) - j(Qy_n - x^*)\| + \alpha_n M_1 + \|x_n + Qy_n\| \cdot \|p_n - x^*\|)$ and observe that $\lambda_n \to 0$ as $n \to \infty$. Then there exists a positive integer N_4 such that $\lambda_n \le \frac{\epsilon}{4}\psi(\frac{\epsilon}{2})$, $\alpha_n \le \epsilon/8(M_2 + M_3 + M_4)$, for all $n \ge N_4$ where $M_2 := \sup\{\|AQy_n\| : n \ge 0\}$, $M_3 := \sup\{\|x_n\| + 2\|Ax_n\| + m : n \ge 0\}$, and M_4 is a positive constant such that

$$\|U_n\| \le c_n M_4 \le \alpha_n^2 M_4.$$

As $\lim_{j\to\infty} n_j = \infty$, we can choose j_* such that $N_{j_*} \geq \max\{n_{j_0}, N_4\}$ so that

$$\|x_{n_{j_*}} - x^*\| < \epsilon,$$

and

$$\lambda_n \leq \frac{\epsilon}{4}\psi\left(\frac{\epsilon}{2}\right), \qquad \alpha_n, \beta_n \leq \frac{\epsilon}{8(M_2 + M_3 + M_4)} \qquad \text{for all } n \geq N_{j_*}. \tag{27}$$

As in the proof of Theorem 1, it follows that $\|x_{n_{j_*+p}} - x^*\| < \epsilon$ for all positive integers $p \geq 1$. Uniqueness also follows as in Theorem 1. This completes the proof. ∎

THEOREM 3. *Let X be a real Banach space which is uniformly convex and uniformly smooth. Let $T\colon D(T) \subseteq X \to X$ be a bounded pseudocontractive operator with closed domain $D(T)$ such that range $R(2I - T)(D(T)) = X$. Let $N(I - T) \neq \phi$. Let $\{u_n\}, \{v_n\}$ be bounded sequences in $D(T)$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ be real sequences satisfying conditions as in Theorem 2 but with Au replaced by $(I - T)u$. Then the sequence $\{x_n\}$ generated from arbitrary x_0, v_0, u_0 in $D(T)$ by*

$$\begin{aligned} x_{n+1} &= Qp_n, \\ p_n &= a_n x_n + b_n TQy_n + c_n u_n, \\ y_n &= a'_n x_n + b'_n Tx_n + c'_n v_n, \qquad n \geq 0, \end{aligned}$$

converges strongly to the unique fixed point x^* of T if there exists a strictly increasing and surjective function $\psi\colon \Re^+ \to \Re^+$ with $\psi(0) = 0$ such that

$$\begin{aligned} \left\langle Tx_{m_0} - Tx^*, j(x_{m_0} - x^*)\right\rangle &\leq \|x_{m_0} - x^*\|^2 - \psi\left(\|x_{m_0} - x^*\|\right)\|x_{m_0} - x^*\| \\ \left\langle TQy_n - Tx^*, j(Qy_n - x^*)\right\rangle &\leq \|Qy_n - x^*\|^2 - \psi\left(\|Qy_n - x^*\|\right)\|Qy_n - x^*\|, \end{aligned} \tag{28}$$

where m_0 is the smallest natural number such that $Tx_{m_0} \neq x_{m_0}$.

Proof. Clearly, from (28), $Gx := x - Tx$ satisfies

$$\begin{aligned} \left\langle Gx_{m_0} - Gx^*, j(x_{m_0} - x^*)\right\rangle &\geq \psi\left(\|x_{m_0} - x^*\|\right)\|x_{m_0} - x^*\|, \\ \left\langle GQy_n - Gx^*, j(Qy_n - x^*)\right\rangle &\geq \psi\left(\|Qy_n - x^*\|\right)\|Qy_n - x^*\|. \end{aligned}$$

Thus by Theorem 2, $\{x_n\}$ converges to the unique solution x^* of the equation $Gx = 0$, which is the unique fixed point of T. This completes the proof. ∎

3.2. *Self-Maps*

If in Theorems 1–3, the domain of the operator is X (i.e., the operator is a self-map) the use of the projection operator Q will not be necessary and X need not be uniformly convex. In fact, the following corollaries follow trivially. In Corollary 4 (below), the Q in the definition of Condition (I) is replaced with I, the identity map on X.

COROLLARY 4. *Let X be a real uniformly smooth Banach space. Let A: $D(A) = X \to X$ be a bounded accretive operator which satisfies condition* (I). *Then there exists a constant $d_0 > 0$ such that for bounded sequences $\{u_n\}, \{v_n\}$ in $D(A)$ and real sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ satisfying the following conditions*:

(i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n,\ n \geq 0$,

(ii) $0 < b_n + c_n \leq d_0,\ 0 \leq b'_n + c'_n \leq d_0,\ n \geq 0$,

(iii) $\sum_{n=0}^{\infty} b_n = \infty,\ c'_n \leq c_n \leq \alpha_n^2$, *where* $\alpha_n = b_n + c_n,\ n \geq 0$,

(iv) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} c'_n = 0$,

the sequence $\{x_n\}$ generated from arbitrary x_0, v_0, u_0 in $D(A)$ by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n (I - A) y_n + c_n u_n, \\ y_n &= a'_n x_n + b'_n x_n + c'_n v_n, \qquad n \geq 0, \end{aligned} \tag{29}$$

converges strongly to the unique solution x^* of the equation $Au = 0$ if and only if there exists a strictly increasing and surjective function ψ: $[0, \infty) := \Re^+ \to \Re^+$ with $\psi(0) = 0$ such that

$$\langle Ay_n - Ax^*, j(y_n - x^*) \rangle \geq \psi(\|y_n - x^*\|)\|y_n - x^*\|.$$

COROLLARY 5. *Let X be a real uniformly smooth Banach space. Let A: $D(A) = X \to X$ be a bounded accretive operator. Let $N(A) \neq \phi$. Then there exists a constant $d_0 > 0$ such that for real sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ satisfying conditions as in Theorem 2 but with Q replaced by I (the identity map on X) and for bounded sequences $\{u_n\}, \{v_n\} \in X$, the sequence $\{x_n\}$ generated from $x_0, u_0, v_0 \in X$ by*

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n (I - A) y_n + c_n u_n, \\ y_n &= a'_n x_n + b'_n (I - A) x_n + c'_n v_n, \qquad n \geq 0, \end{aligned} \tag{30}$$

converges strongly to the unique solution x^* of the equation $Au = 0$ if there exists a strictly increasing and surjective function ψ: $\Re^+ \to \Re^+$ with $\psi(0) = 0$ such that

$$\langle Ax_{m_0} - Ax^*, j(x_{m_0} - x^*) \rangle \geq \psi\left(\|x_{m_0} - x^*\|\right)\|x_{m_0} - x^*\|$$

$$\langle Ay_n - Ax^*, j(y_n - x^*) \rangle \geq \psi(\|y_n - x^*\|)\|y_n - x^*\|,$$

where m_0 is the smallest positive integer such that $Ax_{m_0} \neq 0$.

COROLLARY 6. *Let X be a real uniformly smooth Banach space. Let T: $D(T) = X \to X$ be a bounded pseudocontractive operator. Let $N(I - T) \neq \phi$. Let $\{u_n\}, \{v_n\}$ be bounded sequences in $D(T)$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ be real sequences satisfying conditions as in Theorem 3 but with Q replaced by I (the identity map operator on X). Then the sequence $\{x_n\}$ generated from x_0, v_0, u_0 in $D(T)$ by*

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \\ y_n &= a'_n x_n + b'_n T x_n + c'_n v_n, \qquad n \geq 0, \end{aligned}$$

converges strongly to the unique fixed point x^* of T if there exists a strictly increasing function ψ: $\Re^+ \to \Re^+$, $\psi(0) = 0$ such that

$$\begin{aligned} \left\langle Tx_{m_0} - Tx^*, j(x_{m_0} - x^*)\right\rangle &\leq \|x_{m_0} - x^*\|^2 - \psi\left(\|x_{m_0} - x^*\|\right)\|x_{m_0} - x^*\| \\ \left\langle Ty_n - Tx^*, j(y_n - x^*)\right\rangle &\leq \|y_n - x^*\|^2 - \psi\left(\|y_n - x^*\|\right)\|y_n - x^*\|, \end{aligned}$$

where m_0 is the smallest positive integer such that $Tx_{m_0} \neq x_{m_0}$.

Remark 7. 1. We note that, under the hypotheses of Theorems 1, 2, and 3, and Corollaries 4 and 5, the usual Mann iteration sequence with errors converges strongly to the unique solution x^* of the equation $Au = 0$ or $Gu = 0$. This follows by letting $b'_n = c'_n = 0$.

2. In Corollary 4, if we set $c_n = c'_n = 0$, then the scheme (29) reduces to $x_{n+1} = x_n - \alpha_n A x_n$ which is the so-called steepest descent method considered in [45, 11, 32].

3. In Corollary 5, be setting $c_n = c'_n = 0$, the scheme (30) reduces to

$$\begin{aligned} x_{n+1} &= x_n - \alpha_n A y_n - \alpha_n \beta_n A x_n \\ y_n &= x_n - \beta_n A x_n, \qquad n \geq 0, \end{aligned}$$

which is the Ishikawa-type scheme studied by Z. Haiyun and J. Yuting [25].

4. Thus, our theorems are significant generalizations of the results in [45, 11, 32, 25] and a host of other results to the more general iteration schemes with appropriate error terms. Furthermore, our method of proofs in our more general setting is simpler than the methods used in [11, 45, 25] and is of independent interest.

5. *All out theorems in this paper hold when the mappings are set-valued* if such mappings admit single-valued selections. In such cases each operator in our recursion formula is replaced with its single-valued selection. We omit the details.

6. A prototype for the parameters of our iteration process

$$a_n = a'_n = 1 - \frac{d_0}{(n+1)}$$

$$b_n = b'_n = \frac{d_0}{(n+1)} - \frac{d_0^2}{(n+1)^2}$$

$$c_n = c'_n = \frac{d_0^2}{(n+1)^2}$$

for all integers $n \geq 0$.

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