Fixed Point Theory and Applications: Contributions from Behind Closed Doors

By

Prof. Micah Osilike

1 Introduction

The Vice-chancellor and Chief Executive, Professor Bartho Okolo
The Prochancellor and Chairman of University of Nigeria Governing Council, Dr. Emeka Enejere, KSP
Members of University of Nigeria Governing Council
Deputy Vice-chancellors and other Principal Officers of the University
Deans of Faculties
Former Inaugral Lecturers
Directors of Institutes and Centres
Heads of Departments
Erudite Professors
Heads of Administrative Units
Distinguished Academics and astute Administrators
My lords Spiritual and Temporal
Chiefs and Elders
Great Lions and Lionesses
Ladies and Gentlemen

I am delighted to address you today on my Inaugral Lecture as a Professor of Mathematics, University of Nigeria. This lecture appears over due by all standards since the final pronouncement making me a Professor of Mathematics of this great Univerity was in 2003 with effect from October 1, 1998. However late it seems, today is my turn to deliver it and it is a great day for me.

It is indeed a great day for me because this inaugral lecture provides an opportunity for me to present my research career so far, update colleagues on my current anf future research plans and introduce my research to a wider audience. This is the purpose of inaugral lecture.

Recently when I told one of my close friends that I wanted to deliver this inagural lecture, he laughed and laughed and advised me not to do so because only very few people would understand me. Well, this reflects the contradiction in Mathematics—that a course which is so useful to mankind is irritating to the majority of people in our society. Conceivably, because of the irritation and the underlying phobia, people call us names. Some say we are queer, some say we are weired, others say we are just mad. But Mathematics is an interesting course. It gives a lot of excitement to people there. It is the language of science and the pivot of all subjects both science and art, its benefits are all pervading. Everybody enjoys its benefits at every moment without recognizing it. It is just like the proverbial bee and honey. People hate the bee because it stings but love the sweet honey provided by the bee.

It is against this background of beauty tainted with hatred and phobia that I approach this inaugral lecture. The approach is to ensure that people who ordinarily will be unhappy listening to Mathematics lecture will become at least neutral. I sincerely apologise to those who may still see some of the things I will say as the usual mathematical paranoid as well as to great scientists here who may perceive them as "over simplified"

The Vice-chancellor and Chief Exetive Sir, academic colleagues, lions and lionesses, ladies and gentlemen, my area of Mathematics is Functional Analysis and my main area of research is Operator Theory. In operator theory, I have been working in the area of Fixed Point Theory and Applications.

Let X be a nonempty set and $T : X \to X$ a selfmap. A point $x \in X$ is called a fixed point of T if $Tx = x$. The set of all fixed poinst of T is denoted by $F(T)$ (i.e., $F(T) = \{x \in X : Tx = x\}$). Fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as the properties of the operator $T : X \to X$ in order to obtain results on: the existence (and uniqueness) of fixed points; data dependence of fixed points; the construction of fixed points which includes the study of iteration methods, the stability of the methods, error analysis of the iteration procedures and some sort of comparison of iteration procedures. The underlying spaces X involved in fixed point theory cover a variety of spaces: lattice, metric spaces (complete), normed spaces (complete-Banach spaces), topological vector (linear) spaces, etc. The simplest space is the real line with the usual addition and scalar multiplication and the usual distance measurement. The conditions imposed on the operator T are generally metrical (contractive-type conditions) or compactness type conditions. The theory of fixed points has been revealed as a major, powerful and important tool in the study of nonlinear phenomena. Fixed point theory has been applied in such diverse fields as Differential Equations, Topology, Economics, Biology, Chemistry, Engineering, Game Theory, Physics, Dynamics, Optimal Control, and Functional Analysis. Recent rapid development of efficient techniques for computing fixed points has enormously increased the usefulness of the theory of fixed points for applications. Thus, fixed point theory is increasingly becoming an invaluable tool in the arsenal of the applied mathematics. Many of the most important nonlinear problems of applied Mathematics reduce to solving a given equation which in turn may be reduced to finding the fixed points of a certain operator. Furthermore, contractive-type conditions naturally arise for many of these problems. Thus metrical fixed point theory has developed significantly in the second part of the 20th century. It is in this important and dynamic area that we have been making our contributions for over two decades now. While majority of our contributions are in the area of approximation of fixed points of very important operators and solutions of operator equations, and on the stability of iteration procedures, we have also made modest contributions in the area of existence (and uniqueness) of fixed points of certain operators and solutions of operator equations. In what follows, we present some of our outstanding contributions. In section two, we present results on accretive and pseudocontractive operator equations. Results under this section will include results on approximation of fixed points of strongly pseudocontractive maps, ϕ-strong pseudocontractions, nonexpansive maps, k-strictly pseudocontractive mappings of Browder-Petryshyn-type and demicontractive and hemicontractive maps. We will also present results on approximation of solutions of strongly accretive operator equations, ϕ-strongly accretive operator equations and m-accretive operator equations. Section three will feature results on mappings with "asymptotically-type contractive" behaviours. Results under this chapter will include results on approximation of fixed points of asymptotically nonexpansive mappings, asymptotically pseudocontractive mappings, asymptotically demicontractive mappings and asymptotically ϕ-demicontractive mappings. In section four we present results on approximation of Kpd-operators and discuss other contractive-type mappings which will include our results on quasi-contractive mappings and n-rotative mappings. This section will also feature our results on stability of iteration methods and iteration methods with errors which is substantial. In the final section which is section five, we present our concluding remarks and acknowledgements.

2 Accretive and Pseudocontractive Type Operators

2.1 Introduction

Definition 2.1 A function $\Phi:[0,\infty)\to[0,\infty)$ is said to be a guage function if Φ is continuous and strictly increasing with $\Phi(0)=0$ and $\lim_{t\to\infty}\Phi(t)=\infty$.

Definition 2.2 Let E be a real Banach space. Let E^* denote the topological dual of E and 2^{E^*}

be the collection of all subsets of E^*. Let $\Phi:[0,\infty)\to[0,\infty)$ be a guage function. The mapping $J_\Phi : E \to 2^{E^*}$ defined by

$$J_\Phi x = \{f \in E^* : \langle x, f\rangle = \| x \|\| f \|, \| f \| = \Phi(\| x \|)\}$$

is called duality map with gauge function Φ, where $\langle .,.\rangle$ denotes the generalized duality paring between E and E^*. We note that if $1 < q < \infty$, then $\Phi(t) = t^{q-1}$ is a gauge function. The duality mapping $j_q : E \to 2^{E^*}$ with gauge $\Phi(t) = t^{q-1}$ defined for each $x \in E$ by

$$J_\Phi x = \{f \in E^* : \langle x, f\rangle = \| x \|\| f \|, \| f \| = \| x \|^{q-1}\}$$

is called the *generalised duality mapping*. If $q = 2$, we obtain

$$J_2 := J : E \to 2^{E^*}$$

defined for all $x \in E$ by

$$J_2 x := J(x) = \{f \in E^* : \langle x, f\rangle = \| x \|^2 = \| f \|^2\}.$$

J_2 is known as the normalised duality map. It is well known (see for example [7,21]) that for $1 < q < \infty$, $j_q(x) = \| x \|^{q-2} J(x), \textit{for } x \in X, x \neq 0.$ It follows from Hahn Banach Theorem that if for every gauge function ϕ, $J_\phi x$ is nonempty for any $x \in E$. If E^* is *strictly convex*, then J_q is single-valued and is usually denoted by j_q. If E is *uniformly smooth* (E^* is uniformly convex), then j_q is uniformly continuous on bounded sets.

Lemma 2.1(Kato, [47]): Let E be an arbitrary Banach space. Then for all $x, y \in E$ *and* $\lambda > 0$

$$\| x \| \leq \| x + \lambda y \|$$

if and only if there exists $j(x) \in J(x)$ such that

$$Re\langle y, j(x)\rangle \geq 0.$$

In what follows, we introduce the operators we shall deal with in this section. Throughout, X will denote a real Banach space, $T : D(T) \subseteq X \to R(T) \subseteq X$ an operator with domain $D(T)$ and range $R(T)$ in X, unless otherwise stated.

T is said to be *Lipschitz continuous* with constant $L > 0$ if

$$\| Tx - Ty \| \leq L \| x - y \| \qquad (2.1)$$

for all $x, y \in K$. If $L \in (0,1)$ in (2.1), T is said to be a *strict contraction.* One of the foremost fixed point theorem which has been invaluable in diverse area is the following:

Theorem 2.1 [The Contraction Mapping Principle (The Picard-Banach Caccioppoli Fixed Point Theorem)]. Let (E,ρ) be a *complete metric space* and $T:E\to E$ a *a strict contraction* with contractive constant $k\in[0,1)$. Then

(i) T has a unique fixed point $p\in E$.

(ii) the Picard iteration sequence $\{x_n\}_{n=1}^{\infty}$ associated with T and generated from an arbitrary $x_1\in E$ by

$$x_{n+1}=Tx_n=T^n x_1,\ n\geq 1 \tag{2.2}$$

converges strongly to p

(iii) $\rho(x_n,p)\leq k^{n-1}\rho(x_1,p)$ and $\rho(x_n,p)\leq \dfrac{k}{1-k}\rho(x_n,x_{n-1})\leq \dfrac{k^{n-1}}{1-k}\rho(x_2,x_1)$.

This Theorem which is of fundamental importance in metrical fixed point theory gives existence, uniqueness, and constructive technique for approximation of the fixed point. It dates back to 1890 (E. Picard) but was given the present formulation by S. Banach in 1922 (see also R. Caccioppoli (1930)).

T is said to be *nonexpansive* if $L=1$ in (2.1). Nonepansive mappings are linked intimately with several other nonlinear mappings that are of interest in ordinary and partial differential equations (for example [7,21]).

T is said to be *strongly accretive* if there exists $k>0$ such that for all $x,y\in D(T)$ and for all $j(x-y)\in J(x-y)$, we have

$$\langle Tx-Ty, j(x-y)\rangle \geq k\,\|x-y\|^2\,. \tag{2.3}$$

Without loss of generality, one can always assume $k\in(0,1)$. If $k=0$ in (2.3), i.e., if

$$\langle Tx-Ty, j(x-y)\rangle \geq 0 \tag{2.4}$$

for all $x,y\in D(T), j(x-y)\in J(x-y)$, then T is said to be *accretive*. From Kato's Lemma T is strongly accretive (respectively accretive) if

$$\|x-y\|\leq\|x-y+\lambda[(T-kI)x-(T-kI)y)]\|, \tag{2.5}$$

respectively;

$$\|x-y\|\leq\|x-y+\lambda(Tx-Ty)\|, \tag{2.6}$$

for all $\lambda>0$, where I is the identity operator on $D(T)$.

T is said to be *strongly dissipative (respectively dissipative)* if $-T$ is strongly accretive (respectively accretive).

In Hilbert spaces, j is the identity and $\langle Tx-Ty, j(x-y)\rangle$ reduces to inner product, and strongly accretive operators are called *strongly monotone operators* while accretive operators are called *monotone operators*.

Bruck [14] remarked that the intimate connection between nonexpansive operators and accretive operators accounts partly for the importance of nonexpansive mappings. The class of nonexpansive maps is one of the initial classes of operators for which fixed points results were obtained making use of the geometric structures of the underlying Banach space rather than the compactness properties.

Nonexpansive maps appear in applications as the transition operators for initial value-problems of differential inclusions of the forms $0\in \frac{du}{dt}+T(t)u$, where the operators $\{T(t)\}$ are, in general, set-valued and are accretive or dissipative and minimally continuous.

T is said to be *strictly pseudocontractive* in the terminology of Browder-Petryshyn [9] if for all $x, y\in D(T)$, there exist $\lambda>0$ and $j(x-y)\in J(x-y)$ such that

$$\langle Tx-Ty, j(x-y)\rangle \leq \| x-y \|^2 - \lambda \| x-y-(Tx-Ty) \|^2 . \tag{2.7}$$

Without loss of generality we may assume $\lambda\in(0,1)$. If I denotes the identity mapping on $D(T)$, then (2.7) can be written as

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \lambda \| (I-T)x-(I-T)y \|^2 . \tag{2.8}$$

If $\lambda=0$ in (2.7) (respectively (2.8)), T is said to be *pseudocontractive*. In Hilbert Spaces (2.7) and (2.8) are equivalent to

$$\| Tx-Ty \|^2 \leq \| x-y \|^2 + k \| (I-T)x-(I-T)y \|^2, \tag{2.9}$$

and

$$\| Tx-Ty \|^2 \leq \langle Tx-Ty, x-y\rangle + [1-\frac{(1-k)}{2}] \| (I-T)x-(I-T)y \|^2, \tag{2.10}$$

with $k=(1-2\lambda)<1$. We can assume also $k\geq 0$, so that $k\in[0,1)$. If $\lambda=0$ then $k=1$ in (2.9) and (2.10) and T is pseudocontractive.

T is said to be *demicontractive* (respectively hemicontractive) if

$$F(T)=\{x\in D(T): Tx=x\}\neq\varnothing$$

and the inequalities for strictly pseudocontractive maps (respectively pseudocontractive maps) are satisfied for all $x\in D(T)$ and $y\in F(T)$.

Hicks and Cubicek [44] introduced the class of demicontractive maps in 1977 in Hilbert spaces.

Maruster [63] also considered this class of mappings independently in 1977 as mappings satisfying condition (A).

T is said to be ϕ-*demicontractive* in the terminology of Osilike and Isiogugu [104] if $F(T) \neq \emptyset$ and there exists a strictly increasing continuous function

$\phi:[0,\infty) \to [0,\infty)$ such that $\phi(0) = 0$ and

$$\langle x - Tx, j(x-p) \rangle \geq \phi(\| x - Tx \|), \qquad (2.11)$$

for all $x \in D(T)$, $p \in F(T)$ and $j(x-p) \in J(x-p)$. Every demicontractive map is ϕ-demicontractive with $\phi:[0,\infty) \to [0,\infty)$ given by $\phi(t) = \lambda t^2$. An example of a demicontractive mapping which is not ϕ-demicontractive is given in Osilike and Isiogugu [104].

T is said to be *strongly pseudocontractive* if for all $x, y \in D(T)$ and for all $j(x-y) \in J(x-y)$, there exists $k > 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq k \| x - y \|^2 . \qquad (2.12)$$

If I denotes the identity operator on $D(T)$, then (2.12) is equivalent to

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geq (1-k) \| x - y \|^2 = \lambda \| x - y \|^2 . \qquad (2.13)$$

(2.12) and (2.13) are equivalent to

$$\| x - y \| \leq \| x - y + \sigma[(I - T - \lambda I)x - (I - T - \lambda I)y] \|, \qquad (2.14)$$

for all $\sigma > 0$ This class of mappings and the class of strictly pseudocontractive mappings are independent. It is however also a proper subclass of the class of pseudocontractive maps.

T is said to be ϕ- *strongly pseudocontractive* if there exists a strictly increasing function $\phi:[0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(\| x - y \|) \| x - y \| \qquad (2.15)$$

for all $x, y \in D(T)$. Every strongly accretive operator is ϕ-strongly accretive with $\phi(t) = kt^2$ for some $k \in (0,1)$. A ϕ-strongly accretive operator is accretive.

T is said to be ϕ-strongly pseudocontractive if there exists a strictly increasing function $\phi:[0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \| x - y \|^2 - \phi(\| x - y \|) \| x - y \| \qquad (2.16)$$

for all $x, y \in D(T)$. Every strongly pseudocontractive map is ϕ-strongly pseudocontractive and a ϕ-strongly pseudocontractive map is pseudocontractive.

2.2 Our Results on Approximation of Fixed Points of Strong Pseudocontractions and Solutions of Operator Equations of the (Strongly) Accretive Type

It is easily seen from the definition of a strong pseudocontraction that if it has a fixed point, then the fixed point is unique. If $T: X \to X$ is strongly pseudocontractive and continuous, then T has a unique fixed point (see for example [33]). Furthermore if X is uniformly smooth and $T: X \to X$ is *demicontinuous*, then T has a unique fixed point [33]. From definitions, T is strongly pseudocontractive if and only if $(I-T)$ is strongly accretive. Consequently, if T is strongly accretive, the operator equation

$$Tx = f,\ f \in R(T), \tag{2.17}$$

which has been studied by several authors (see for example [7,21]) has intimate connection with fixed points of pseudocontractions. It is well known (see for example Theorem 13.1 of Deimling [33])) that for any given $f \in X$, the equation (2.17) has a unique solution if $T: X \to X$ is strongly accretive and continuous, or X is uniformly smooth and $T: X \to X$ is strongly accretive and demicontinuos. In general, if (2.17) has a solution, the strong accretivity of T guarantees that the solution is unique. Usually (2.17) is converted to a "fixed point equation" using the auxiliary operator $S: X \to X$ defined by

$$Sx = x - Tx + f. \tag{2.18}$$

Thus x^* is a solution of (2.17) if and only if x^* is a fixed point of S. Furthermore, since T is strongly accretive with constant $k \in (0,1)$, we have

$$\langle Sx - Sy, j(x-y)\rangle \le (1-k)\| x-y \|^2,\ \forall\ x, y \in X,\ and\ \forall\ j(x-y) \in J(x-y). \tag{2.19}$$

Hence S is strongly pseudocontractive. Thus the solution of (2.17) is indeed the fixed point of the mapping S and vice-versa.

An accretive operator T is said to be *m-accretive* if T is accretive and $(I+rT)(D(T)) = X$ (i.e., $(I+rT): D(T) \to X$ is surjective) for all $r > 0$. An important example of m-accretive operator in n-dimensional Euclidean space is $-\Delta$, where Δ denote the Laplacian operator. If T is m-accretive, then for any given $f \in X$ the equation

$$x + Tx = f \tag{2.20}$$

has a unique solution. It is shown in Martin [60-61] that if $T: X \to X$ is accretive and continuous, then

T is m-accretive. Equation (2.20) is usually converted into a "fixed point equation" using the auxiliary operator $S : X \to X$ defined for each $x \in X$ by

$$Sx = f - Tx. \tag{2.21}$$

The operator S is easily shown to be strongly pseudocontractive. Thus the existence (uniqueness) and approximation of fixed points of strong pseudocontractions and the existence (uniqueness) and approximation of equations (2.17) and (2.20) are intimately related.

Many authors (see for example [1-2,7,16,20-31,35,44-46,49-56,58-59,63,65-110,118-128,130-131,137-138]) have applied the *Mann* and *Ishikawa* iteration schemes described below for the iterative approximation of fixed points of strong pseudocontractions and solutions of equations (2.17) and (2.20).

Mann Iteration Scheme (Mann [58]). Let C be a nonempty closed convex subset of X and let $T : C \to C$ be a mapping. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0,1)$ satisfying some appropriate conditions. Then the Mann iteration scheme is generated from an arbitrary $x_1 \in E$ by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \; n \geq 1 \tag{2.22}$$

The Ishikawa Iteration Process (Ishikawa [46]). Let C be a nonempty closed convex subset of X and let $T : C \to C$ be a mapping. Then the Ishikawa iteration process is given by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n Tx_n, \; n \geq 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n, \; n \geq 0 \end{aligned} \tag{2.23}$$

' where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in [0,1] satisfying some suitable conditions. In its original form, we have the conditions: $0 \leq \alpha_n \leq \beta_n \leq 1$

$$\lim_{n\to\infty} \beta_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty,$$

and hence the scheme does not include the Mann iteration scheme as a special case.

We have made notable contributions in the area of approximation of fixed points of strong pseudocontractions and solutions of (1.16) and (1.19) using the above iteration schemes. Our earliest contribution appeared in *Bull. Austra. Math. Soc.* **46 (3)** (1992), 413-424 (MR 1190344 (93i:47091)). This was followed by several outstanding contributions which include:

- Chidume and Osilike, *Numer. Funct. Anal. Optim.* **15 (7-8)**, (1994), 779-790 (MR1305573 (95i:47106))
- Chidume and Osilike, *J. Math. Anal. Appl.* **189 (1)** (1995), 225-239 (MR1312040 (95m:47100))

- Chidume and Osilike, *J. Math. Anal. Appl.* **(192 (3)** (1995), 727-741 (MR1336474 (96i:4709))
- Osilike, *Soochow J. Math.* **22 (4)** (1996), 485-494 (MR1426554 (97i:47122))
- Osilike, *J. Math. Anal. Appl.* **209 (1)** (1997), 20-24 (MR1444508 (98c:47080))
- Osilike, *J. Math. Anal. Appl.* **213 (1)** (1997), 91-105 (MR1469362 (98g:47054))
- Chidume and Osilike *Nonlinear Anal.* **31 (7)**, (1998), 779-789 (MR1488373 (99b:47082))
- Osilike, *Soochow J. Math.* **24 (2)** (1998), 141-148 (MR1654162 (99h:47071))
- Chidume and Osilike, *Nonlinear Anal.* **36 (7)**, (1999), 863-872 (MR1682844 (2000h:47082))
- Osilike, *Indian J. Pure Appl. Math.* **31 (2)**, 117-127 (MR1743109 (2000j:47102))
- Osilike and Igbokwe, *Comput. Math. Appl.* **40 (4-5)**, 291-300 (MR1772655 (2001b:47123))
- Chidume and Osilike, *Proc. Edinb. Math. Soc. (2)* **44 (1)**, 187-199 (MR1879218 (2002i:47067))

Some of our notable results which have remained the most general result in the literature from the above results include:

Theorem 2.2 ([28-29]). Let X be an arbitrary real Banach space and K a nonempty closed convex subset of X. Let $T: K \to K$ be a uniformly continuous strong pseudocontraction and let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be real sequences satisfying the following conditions:

(i) $0 \le \alpha_n, \beta_n \le 1$ (ii) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and (iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in X$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,\ n \ge 1$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n,\ n \ge 1$$

converges strongly to the fixed point of T.

Corollaries of Theorem 2.2 include the strong convergence results for the operator equation (2.17) when T is strongly accretive and uniformly continuous, and the operator equation (2.20) when T is accretive and uniformly continuous.

Theorem 2.3 ([25-26]). Let E be a real uniformly smooth Banach space and let $T: E \to E$ be a strongly accretive and demicontinuous operator. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be as in Theorem 1.2. Then for any $f \in E$ the Ishikawa iteration sequence (in Theorem 1.2) converges strongly to the solution of the equation $Tx = f$.

The following Theorem has also remained a result of independent interest since its appearance in item five above.

Theorem 2.4 ([78]). Let E be a real reflexive Banach space, and $T: D(T) \subseteq E \to E$ an m-accretive and locally l-Lipchitz operator. Let $D(T)$ be open and let $x^* \in D(T)$ be the unique solution of the equation $x + Tx = f$, $f \in E$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:

(i) $0 \le \alpha_n \le \dfrac{1}{2[L^2 + 2L + 2]}, n \ge 1,$

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty.$

Then there exists a closed convex neighbourhood B of x^* contained in $D(T)$ and for any given $x_0 \in B$, a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of B such that on setting

$$p_n = (1 - \alpha_n)x_n + \alpha_n(f - Tx_n), n \ge 1,$$

the sequence $\{p_n\}_{n=1}^{\infty}$ satisfies the condition

$$\| p_{n-2} - x_n \| = \inf\{\| p_{n-1} - x \| : x \in B\} \ \forall\ n \ge 2,$$

and converges strongly to x^*. Moreover, if $\alpha_n = \dfrac{1}{2[L^2 + 2L + 2]}$ for all $n \ge 1$, then

$$\| p_n - x^* \| \le \rho^n \| p_1 - x^* \|,$$

where $\rho = (1 - \dfrac{1}{4[L^2 + 2L + 2])} \in (0,1).$

2.3 Our Results on ϕ-strong Pseudocontractions and ϕ-Strongly Accretive Operator Equations

We first studied the class of ϕ-strongly accretive operators and ϕ-strong pseudocontractions in q- *uniformly smooth Banach spaces* in 1996 (see Osilike [74] *J. Math. Anal. Appl.* **(200 (2)** (1996), 259-271 (MR1391148 (97d:65032))). In the results the author provided an example of a ϕ-strongly accretive operator which is not strongly accretive (and hence an example of a ϕ-strong pseudocontraction which is not a strong pseudocontraction). The example has been cited by several authors. Several other notable results which followed the above cited result include:

- Osilike, *J. Math. Anal. Appl.* **227 (2)** (1998), 319-334 (MR1654162 (99h:47071))
- Osilike, *Nonlinear Anal.* **36 (1)** (1999), 1-9 (MR1670303 (2000j:47103))

- Osilike, *Fixed Point Theory and Applications* (Chinju, 1998), 227-236 Nova sci. Publ., Huntington, NY, 2000 (MR1761229 (2001b:47122))
- Osilike, *Nonlinear Analysis* **42 (2)** (2000), 291-300 (MR1773985 (2003a:47130))

Some of the results in the cited references include:

Theorem 2.5 ([85]). Suppose X is an arbitrary real Banach space and $T: X \to X$ a Lipschitz ϕ-strongly accretive operator. Suppose the equation $Tx = f$ has a solution and suppose $\{\alpha\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences satisfying the following conditions:

(i) $0 \le \alpha_n, \beta_n \le 1$ (ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$, (iii) $\sum_{n=1}^{\infty}\alpha_n^2 < \infty$, and (iv) $\sum_{n=1}^{\infty}\alpha_n\beta_n < \infty$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in X$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f + (I-T)y_n),\ n \ge 1$$

$$y_n = (1-\beta_n)x_n + \beta_n(f + (I-T)x_n),\ n \ge 1$$

converges strongly to the solution of the equation $Tx = f$.

The result (item 2 above) which contains Theorem 1.5 also contains a similar theorem for the iterative approximation of fixed points of ϕ-pseudocontractions.

In the course of proving Theorem 1.5 and other related results, we showed that for $Sx = f + (I-T)x$ where T is ϕ-strongly accretive, we have

$$\langle (I-S)x - (I-S)y, j(x-y)\rangle \ge \sigma(x,y)\,\| x-y \|^2, \tag{2.24}$$

where $\sigma(x,y) = \dfrac{\phi(\| x-y \|)}{1+\phi(\| x-y \|)+\| x-y \|} \in [0,1), \forall\, x, y \in X$. Using (2.24) and Kato's result we also obtained that

$$\| x-y \| \le \| x-y+r[(I-S)x-\sigma(x,y)x-((I-S)y-\sigma(x,y)y)] \|, \tag{2.25}$$

for all $x, y \in X$ and for all $r > 0$. Inequalities (2.24) and (2.25) hold for any ϕ-strong pseudocontraction S and they have been used by several authors since its appearance in our work.

We also proved the following Lemma:

Lemma 2.2 ([85]). Suppose $\phi: [0,\infty) \to [0,\infty)$ is a strictly increasing function with $\phi(0) = 0$. Suppose $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ are sequences of nonnegative numbers satisfying the following conditions:

(i) $\sum_{n=1}^{\infty}\lambda_n = \infty$,

(ii) $\sum_{n=1}^{\infty}\delta_n < \infty$.

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of nonnegative numbers satisfying the inequality:

$$a_{n+1} \leq [1+\delta]a_n - \lambda_n \frac{\phi(a_{n+1})}{1+\phi(a_{n+1})+a_{n+1}}, n \geq 1.$$

Then $\lim_{n\to\infty} a_n = 0.$

Lemma 2.2 have been employed by several authors in the prove of convergence theorems for operators of the ϕ-accretive and ϕ-pseudocontractive type.

2.4 Our Results on Nonexpansive mappings, Strictly Pseudocontractive Mappings of Browder-Petryshyn Type, Demicontractive Mappings, Hemicontractive Mappings and ϕ-Demicontractive Mappings

Our foremost results in this area include:

Osilike and Igbokwe, *Comput. Math. Appl.* **40 (4-5)**, 291-300 (MR1772655 (2001b:47123)), and

Osilike and Udomene, *J. Math. Anal. Appl.* **256 (2)**, 431-445 (MR1821748 (2002b:47118)).

In Osilike and Igbokwe we obtained the following:

Theorem 2.6 ([91]). Let H be real Hilbert space anf C a nonempty closed convex subset of H. Let $T: C \to C$ be an L-Lipschitz hemicontractive mapping, and let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}, \{a_{n'}\}_{n=1}^{\infty}$ $\{b_{n'}\}_{n=1}^{\infty}$ and $\{c_{n'}\}_{n=1}^{\infty}$ be real sequences in $[0,1]$ satisfying the conditions:

(i) $a_n + b_n + c_n = a_{n'} + b_{n'} + c_{n'} = 1, n \geq 1$

(ii) $0 < \varepsilon \leq b_{n'} \leq b < 1, \forall\, n \geq 1$, for some $\varepsilon > 0$ and for some $b \in (0, \frac{1}{[(\sqrt{1+L^2})+1]}),$

(iii) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} c_{n'} < \infty$. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be bounded sequences in C and let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = a_{n'}x_n + b_{n'}Ty_n + c_{n'}v_n, n \geq 1$$

$$y_n = a_n x_n + b_n Tx_n + c_n u_n, n \geq 1.$$

Then $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$.

One of the consequences of Theorem 1.6 is that if $T: H \to H$ is an L-Lipschitz pseudocontractive mapping with a nonempty fixed-point set and $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}, \{a_{n'}\}_{n=1}^{\infty}$ $\{b_{n'}\}_{n=1}^{\infty}$, $\{c_{n'}\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are as in Theorem 2.6, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to a fixed point of T.

It also follows from Theorem 2.6 that if $T: H \to H$ is an L-Lipschitzian monotone operator and $f \in R(T)$, then if $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}, \{a_{n'}\}_{n=1}^{\infty}$ $\{b_{n'}\}_{n=1}^{\infty}$, $\{c_{n'}\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are as in Theorem 1.6, we have that the sequence $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in H$ by

$$x_{n+1} = a_{n'}x_n + b_{n'}Sy_n + c_{n'}v_n,\ n \geq 1$$

$$y_n = a_n x_n + b_n S x_n + c_n u_n,\ n \geq 1$$

has the property that $\lim_{n\to\infty} \| x_n - Sx_n \| = 0$ and $\{x_n\}_{n=1}^{\infty}$ converges weakly to a solution of the equation $Tx = f$.

In the course of the proofs of the results in Osilike and Igbokwe, we proved the following results which are of independent interest and which have been employed by several authors.

Lemma 2.3 ([91]). Let E be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\| \alpha x + \beta y + \gamma z \|^2 = \alpha \| x \|^2 + \beta \| y \|^2 + \gamma \| z \|^2 - \alpha\beta \| x - y \|^2 - \alpha\gamma \| x - z \|^2 - \beta\gamma \| y - z \|^2 . \quad (2.26)$$

If $\gamma = 0$ in (1.24) we obtain a famous identity developed by Ishikawa [38] which has been very useful in Hilbert spaces.

Lemma 2.4 ([91]) Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n,\ n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

In Osilike and Udomene we proved among other important results of independent interest that if E is a real 2-*uniformly smooth Banach space* which is also *uniformly convex*, K a nonempty closed convex subset of E, and T a strictly pseudocontractive self-map of K in the sense of F. E. Browder and W. V. Petryshyn [J. Math. Anal. Appl. 20 (1967), 197–228; MR0217658 (36 #747)], then $(I - T)$ is demiclosed at zero. We further proved weak and strong convergence theorems for the iterative approximation of fixed points of T using the Mann and Ishikawa iteration methods when T has at least one fixed point. Results of Browder and Petryshyn [J. Math. Anal. Appl. 20 (1967), 197–228; MR0217658 (36 #747)] and B. E. Rhoades [Trans. Amer. Math. Soc. 196 (1974), 161–176; MR0348565 (50 #1063)] are obtained as special cases. Existence of a fixed point of strictly pseudocontractive self-map of nonempty closed convex and bounded subset of a 2-uniformly smooth Banach space which is also uniformly convex also followed from our result.

The reviewer of the article in AMS Mathematical Review—Professor S.L. Singh ended his review with the remark "This well-researched article will be useful to research mathematicians." It has certainly been so and the work has been cited by several authors.

Several other contributions under this heading which we can not discuss in detail include:

- Osilike, *J. Nigerian Math.* Soc. **12** (1993), 73-79, (MR1328788 (96f:47126)).
- Osilike, *Bull. Korean Math. Soc.* **37 (1)** (2000), 153-169, (MR1752204 (2002a:47103)).
- Osilike, *Bull. Korean Math. Soc.* **37 (1)** (2000), 153-169, (MR1752204 (2002a:47103)).
- Osilike, *J. Math. Anal. Appl.* **294 (1)** (2004), 73-81, (MR2059789 (2005b:47134)).
- Osilike, *Panamer. Math. J.* **14 (3)** (2004), 89-98 (MR2070244 (2005b:47135)).
- Osilike, Isiogugu,and Nwokoro, *Fixed Point Theory Appl.* 2007, Art. ID 64306, 7 pp, (MR2377371 (2008j:47051)).
- Osilike, Isiogugu,and Nwokoro, *J. Nigerian Math. Soc.* **27** (2008), 91-108, (MR2421797 (2009g:47147)).
- Osilike, and Shehu, *Nonlinear Anal.* **70 (10)** (2009), 3575-3583, (MR2502766 (2010e:47134)).
- Udomene, Osilike, *J. Nigerian Math. Soc.* **28** (2009), 77-95, MR2536876 ((2010i:47133)).

3 Mappings with Asymptotically Type Contractive Behaviours

3.1 Introduction

In the following, K is a nonempty subset of a real Banach space X .

A mapping $T: K \to K$ is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that

$$\| T^n x - T^n y \| \leq L \| x - y \|, \forall\ x, y \in K. \qquad (3.1)$$

.

T is said to be *asymptotically nonexpansive* (see for example [39]) if $\forall\ x, y \in K, \exists$ a sequence

$\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty), \lim_{n\to\infty} k_n = 1$ such that

$$\| T^n x - T^n y \| \le k_n \| x - y \| . \qquad (3.2)$$

Every Asymptotically nonexpanve mapping is uniformly L-Lipschitzian and the class of asymptotically nonexpansive mappings contains the important class of nonexpansive mappings as a proper subclass.

T is said to be *k-strictly asymptotically pseudocontractive* with sequences $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ (see for example [118]) if for all $x, y \in K$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n \| x-y \|^2 - k \| (x - T^n)x - (y - T^n)y \|^2 \qquad (3.3)$$

for all $n \in N$. T is called *asymptotically demicontractive* with sequences $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ (see [118]) if $F(T) = \{x \in K : Tx = x\} \ne \varnothing$ and for all $x \in K$, $p \in F(T)$, there exists $j(x-p) \in J(x-p)$ such that

$$\langle T^n x - p, j(x-p)\rangle \le k_n \| x-p \|^2 - k \| x - T^n \|^2 . \qquad (3.4)$$

These classes of operators were first studied in Hilbert spaces by Qihou [118]. In ([83], 1998) we proved that in Hilbert spaces (3.3) and (3.4) are equivalent to

$$\| T^n x - T^n y \|^2 \le k_n \| x-y \|^2 + (1-k) \| (I - T^n)x - (I - T^n)y \|^2, \forall\ x, y \in K \quad (3.5)$$

and

$$\| T^n x - p \|^2 \le k_n \| x-p \|^2 + (1-k) \| x - T^n x \|^2, \forall\ x \in K \text{ and } \forall\ p \in F(T). \quad (3.6)$$

respectively which are the inequalities considered by Qihou [118]. In [83] we also showed that in arbitrary real Banach spaces, every k-strictly asymptotically pseudocontractive mapping is uniformly L-lipschitzian.

It is obvious that a k-strictly asymptotically pseudocontractive mapping T with a nonempty fixed point set $F(T)$ is also asymptotically demicontractive. An example in ([83],[118]) shows that the class of asymptotically demicontractive mappings is more general than the class of k-strictly asymptotically pseudocontractive mappings.

T is said to be *asymptotically pseudocontractive* if $k = 0$ in (3.3), and *asymptotically hemicontractive* if $k = 0$ in (3.4).Thus in Hilbert spaces T is asymptotically pseudocontractive (respectively asymptotically hemicontractive) if (3.5) (respectively (3.6)) are satisfied for $k = 0$.

T is said to be *asymptotically ϕ-pseudocontractive* with sequence $\{k_n\} \subseteq [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ if there exists a strictly increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \le k_n \| x-y \|^2 - \phi(\| x - T^n x - (y - T^n y) \|), \qquad (3.7)$$

for all $x, y \in K$, $j(x-y) \in J(x-y)$, and for all $n \in IN$.

T is *asymptotically ϕ-demicontractive* in the terminology of Osilike and Isiogugu [104] if $F(T) \neq \emptyset$ and for all $x \in K$, $p \in F(T)$ there exist $j(x-p) \in J(x-p)$ and a strictly increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - p, j(x-p)\rangle \le k_n \| x-p \|^2 - \phi(\| x - T^n x \|). \qquad (3.8)$$

Every asymptotically demicontractive map is asymptotically ϕ-demicontractive with $\phi : [0, \infty) \to [0, \infty)$ given by $\phi(t) = 12(1-k)t^2$. An example given in [104] by Osilike and Isiogugu shows that the class of asymptotically ϕ-demicontractive maps is more general than the class of asymptotically demicontractive maps.

For the approximation of fixed points of mappings with asymptotically type contractive definitions, the following averaging iteration methods (modified Mann and Ishikawa Iteration methods) introduced by Schu [123,124] have been applied extensively by various authors.

Averaging (Modified) Mann Iteration Scheme (Sch [123,124], (1991)). Let $T : K \to K$ be a given map. The modified Mann iteration is generated from an arbitrary $x_1 \in K$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, n \ge 1,$$

where $\{\alpha_n\}$ is a suitable sequence in $[0,1]$.

Averaging (Modified) Ishikawa Iteration Scheme ([118]). This is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, n \ge 1$$

$$y_n = (1-\beta_n)x_n + \beta_n T^n x_n, n \ge 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequences in $[0,1]$.

3.2 Our Results on Mappings with With Asymptotically Type Contractive Behaviours

We have made extensive contributions in this area. Some of our major results which have remained most general include

- Osilike, M.O.; Aniagbosor, S. C. *Math. Comput. Modelling* **32 (10)** (2000), 1181-1191, (MR1791754 (2001h:47089)).
- Osilike, M.O.; Isiogugu, F. O. *Panamer. Math. J.* **15(3)** (2005), 59-67, (MR2144195 (2005k:47124)).
- Osilike, M.O.; Udomene, A.; Igbokwe, D. I.; Akuchu, B. G. *J. Math. Anal. Appl.* **326 (2)** (2007), 1334-1345, (MR2280984 (2007m:47113)). In Osilike and Aniagbosor, we proved among other things:

Theorem 3.1 ([93]). Let E be a real Banach space and let C be a nonempty closed convex subset of E. Let $T:C\to C$ be an asymptotically nonexpansive mapping with $F(T)=\{x\in C:Tx=x\}\neq\varnothing$ and $\{k_n\}_{n=1}^{\infty}\subseteq[1,\infty)$ a sequence such that $\lim_{n\to\infty}k_n=1$. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be bounded sequence in C and let $\{a_n\}_{n=1}^{\infty},\{b_n\}_{n=1}^{\infty},\{c_n\}_{n=1}^{\infty},\{a_{n'}\}_{n=1}^{\infty}$ $\{b_{n'}\}_{n=1}^{\infty}$ and $\{c_{n'}\}_{n=1}^{\infty}$ be real sequences in $[0,1]$ satisfying the conditions:

(i) $a_n+b_n+c_n=a_{n'}+b_{n'}+c_{n'}=1,\, n\geq 1$

(ii) $0<a\leq b_{n'}\leq b<1,\ \forall\, n\geq 1$, for some $a,b\in(0,1)$,

(iii) $\lim_{n\to\infty}b_n=0$

(iv) $\sum_{n=1}^{\infty}c_n<\infty$, $\sum_{n=1}^{\infty}c_{n'}<\infty$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1\in C$ by

$$x_{n+1}=a_{n'}x_n+b_{n'}Ty_n+c_{n'}v_n,\, n\geq 1$$
$$y_n=a_nx_n+b_nTx_n+c_nu_n,\, n\geq 1.$$

Then: (A) if E satisfies Opial's condition, the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to $p\in F(T)$; (B) if T^m is compact for some $m\geq 1$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $q\in F(T)$.

The reviewer of this article remarked that "this theorem extends many well-known previous results".

In Osilike and Isiogugu [104] we introduced the class of ϕ-demicontractions and provided an example to show that it is more general than the class of demicontractive maps which has been studied by several authors. We further proved the following;

Theorem 3.2 ([104]). Let E be a real Banach space and K a nonempty convex subset of E. Let $T:K\to K$ be a uniformly L-Lipschitzian asymptotically ϕ-demicontractive mapping with a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying:

(i) $0 < \alpha_n < 1$

(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$

(iii) $\sum_{n=1}^{\infty}\alpha_n^2 < \infty$.

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 1.$$

Then $\liminf_{n\to\infty} \| x_n - Tx_n \| = 0$.

Theorem 3.1 yielded several far-reaching weak and strong convergence results which are extensions of many well-known previous results.

In Osilike,Udomene, Igbokwe and Akuchu we showed that the class of strictly pseudocontractive maps and the class of asymptotically strictly pseudocontractive maps are independent. We further proved among other important results the following:

Theorem 3.3 ([105]). Let E be a real 2-uniformly smooth Banach space which is also uniformly convex. Let C be a nonempty closed convex subset of E and $T: C \to C$ a k-strictly asymptotically pseudocontractive mapping with a nonempty fixed point set. Then $(I - T)$ is demiclosed at zero. If C is in addition bounded, then T has a fixed point.

Theorem 9.4 ([105]). Let E be a real 2-uniformly smooth Banach space and let C be a nonempty convex subset of E. Let $T: C \to C$ be a k-strictly asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and let $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the conditions:

(i) $0 \leq \alpha_n \leq 1,\ n \geq 1,$

(ii) $0 < a \leq \alpha_n \leq b < 2(1-k)2c_2(1+L),\ n \geq 1,$

Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n,\ n \geq 1.$$

Then

(a) $\lim_{n\to\infty} \| x_n - x^* \|$ exists for all $x^* \in F(T)$

(b) $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$.

Theorem 3.4 also yielded several weak and strong convergence results for asymptotically strictly pseudocontractive maps.

Other various contributions we have made in this area include the results presented in the following articles.

- Osilike, M.O. *Indian J. Pure Appl. Math.* **29 (12)** (1998), 1291-1300, (MR1671343 (99m:47066)).
- Osilike, M. O.; Aniagbosor, S. C. *Indian J. Pure Appl. Math.* **32 (10)** (2001), 1519-1537, MR1878067 ((2002i:47079)).
- Osilike, M. O.; Igbokwe, D. I. *Fixed point theory and applications.* **Vol. 2** (Chinju/Masan, 2000), 27-42, Nova Sci. Publ., Huntington, NY, 2001, (MR2043179 (2004m:47140)).
- Osilike, M. O.; Aniagbosor, S. C.; Akuchu, B. G. *Panamer. Math. J.* **12 (2)** (2002), 77-88 (MR1895772 (2003c:47098)).
- Osilike, M. O.; Igbokwe, D. I. *Bull. Korean Math. Soc.* **39 (3)** (2002), 389-399, (MR1920697 (2003g:47091)).
- Osilike, M. O.; Akuchu, B. G. *Fixed Point Theory Appl.* 2004, no. 2, 81-88, (MR2086707 (2005c:47087)).
- Osilike, M. O.; Shehu, Y. *Comput. Math. Appl.* **57 (9)** (2009), 1502-1510, (MR2509963 (2010c:47170)).
- Osilike, M. O.; Shehu, Y. *Appl. Math. Comput.* **213 (2)** (2009), 548-553, (MR2536680).

4 Kpd Operators, Other contractive-type Mappings, and Stability of iteration Methods and Iteration Methods with Errors

Introduction

In this section we present summaries of results on approximation of Kpd-operators and discuss other contractive-type mappings which will include our results on quasi-contractive mappings and n-rotative mappings. This chapter will also feature our results on stability of iteration methods and iteration methods with errors which is substantial.

Quasi-contractive Maps Let C be a nonempty subset of X, a mapping $T : C \to C$ is said to be *quasi-contractive* (see e.g., [120]) if there exists a constant $k \in [0,1)$ such that, (1)

$$\| Tx - Ty \| \le k \max\{\| x - y \|, \| x - Tx \|, \| y - Ty \|, \| x - Ty \|, \| y - Tx \|\} \qquad (4.1)$$

for all $x, y \in K$. In [120], Rhoades showed that the contractive definition (1), apart from being an obvious generalization of the well-known contraction mapping, is one of the most general contractive definitions for which Picard iterations give a unique fixed point.

Chidume and Osilike, *Bull. Korean Math. Soc.* **30 (2)** (1993), 201–212, (MR1239289 (94h:47109))) studied quasi-contractive maps $T : C \to C$ and proved that the Ishikawa and Mann iterates satisfying certain appropriate conditions converge strongly to the unique fixed point of T, when X is a real uniformly smooth Banach space with modulus of smoothness of power type $q > 1$.

Osilike, *Indian J. Pure Appl. Math.* **28 (9)** (1997), 1251-1265, (MR1605502 (98m:47090)) also proved stability results for the Mann and Ishikawa Iteration schemes for quasi-contractive maps.

K**-Positive Definite** (Kpd) **Operators.** In [31] Chidume and Aneke extended the notion of K -positive definite (Kpd) operators of Martyniuk [62] and Petryshyn ([115-116]) from Hilbert spaces to arbitrary real Banach spaces. They called a linear unbounded operator A defined on a dense domain $D(A)$ in X a Kpd operator if there exist a continuously $D(A)$-invertible closed linear operator K with $D(A) \subseteq D(K)$ and a constant $c > 0$ such that for $j(Kx) \in J(Kx)$,

$$\langle Ax, j(Kx) \rangle \ge c \| Kx \|^2, \forall x \in D(A). \qquad (4.2)$$

In Hilbert spaces, j is the identity and the inequality (3.1) coincides with one studied in Hilbert spaces in ([62],[115-116]).

Our major contribution in area of approximation of Kpd operators which has remained the most general result appeared in:

Osilike and Udomene, *Bull. Korean Math. Soc.* **38 (2)** (2001), 231–236, (MR1827675 (2002b:47021) 47A50)). In this article we constructed a sequence of Picard iterates suitable for the approximation of solutions of K-positive definite operator equations in arbitrary real Banach spaces. We obtained explicit error estimate and showed that convergence is as fast as a geometric progression.

Our other contributions on Kpd operators appeared in

Chidume and Osilike, *J. Math. Anal. Appl.* **210 (1)** (1997) 1-7, MR1449506 ((98c:47010)).

k**-Lipschitzian and Firmly** k**-Lipschizian** 2**-Rotative Maps.** Let C be a nonempty subset of X, a mapping $T : C \to C$ is said to be *firmly* k*-Lipschitzian* (see e.g., [101]) if there exists a constant $k \in [0,1)$ such that,

$$\| Tx - Ty \| \le \| k(1-t)(x-y) + t(Tx-Ty) \| \qquad (4.3)$$

for all $x, y \in C$ and for all $t \in [0,1]$. The class of firmly k-Lipschitzian mapping is a proper subclass of the class of k-Lipschitzian mappings.

T is said to be n- *rotative with constant* a if there exist a positive constant a and a positive integer n such that

$$\| T^n x - x \| \le a \| Tx - x \| \qquad (4.4)$$

for all $x \in C$. We first studied this class of operators in our M.Sc. project report where we made contributions in the existence of fixed points of certain class of of the operators in $L_p(\text{or } \ell_p)$ spaces, $1 < p < \infty$.

Other contributions appeared in:

Osilike and Akuchu, *Nova Sci. Publ., Hauppauge, NY,* 2004, (MR2067310 (2005b:47133)). In the article we proved existence results for fixed points of k-Lipschitzian and firmly k-Lipschizian 2-rotative maps $T : C \subseteq E \to C$, where C is a nonempty closed convex subset of a q-uniformly smooth Banach space E, $1 < q < \infty$.

Stability of Iteration Methods and Iteration Methods With Errors. Let T be a self-map of X. Let $x_0 \in X$ and let $x_{n+1} = f(T, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=1}^{\infty}$ in X. For example, the function iteration $x_{n+1} = f(T, x_n) = Tx_n$.

Let $F(T) = \{x \in X : Tx = x\} \ne \emptyset$ and let $\{x_n\}_{n=1}^{\infty}$ converge strongly to $p \in F(T)$. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in X and let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence in $\Re^+ = [0, \infty)$ given by $\| y_{n+1} - f(T, y_n) \|$. If $\lim\limits_{n\to\infty} \varepsilon_n = 0$ implies that $\lim\limits_{n\to\infty} y_n = p$, then the iteration procedure defined by $x_{n+1} = f(T, x_n)$ is said to be T- *stable* or *stable with respect to* T (see for example ([2],[41-43],[50-56],[71],[73],[75-76],[80-81],[84],[92],[138]). The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be *almost* T- *stable* or *almost stable with respect to* T if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ implies that $\lim\limits_{n\to\infty} y_n = p$ (see for example [84]). It is shown in [84] that a T-stable iteration procedure is also almost T-stable, while some almost T-stable iteration methods may fail to be T-stable. Stability of iteration procedures for different classes of mappings have been studied by several authors (see for example [2],[41-43],[50-56],[71],[73],[75-76],[80-81],[84],[92],[138]).

In particular, the stability of the Mann iteration procedure $M(T, x_n, \alpha_n)$ given by

$$x_{n+1} = f(T, x_n) = (1-\alpha_n)x_n + \alpha_n Tx_n, x_1 \in X, n \geq 1,$$

and the Ishikawa iteration procedure $I(T, x_n, \alpha_n, \beta_n)$ given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n], x_1 \in X, n \geq 1,$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are suitable sequences in $[0,1]$, had been studied extensively by various authors for various classes of nonlinear mappings (see for example [2],[41-43],[50-56],[71],[73],[75-76],[80-81],[84],[92],[138]). Apart from contractive-type mappings, other classes of mappings that has been studied extensively are the classes of strong pseudocontractions and ϕ-strong pseudocontractions.

In [49], Liu introduced the Mann iteration method with error $M_e(T, x_n, \alpha_n, u_n)$ given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n + u_n,\ x_1 \in X,\ n \geq 1,$$

and the Ishikawa iteration method with errors $I_e(T, x_n, \alpha_n, \beta_n, u_n, v_n)$ given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n + v_n] + u_n,\ x_1 \in X,\ n \geq 1,$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are suitable sequences in $[0,1]$ and $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are summable sequences in X (i.e., $\sum_{n=1}^{\infty} \| u_n \| < \infty$, and $\sum_{n=1}^{\infty} \| v_n \| < \infty$). Xu [131], introduced a modified Mann iteration method with error $M_{me}(T, x_n, a_n, b_n, c_n, u_n)$ given my

$$x_{n+1} = a_n x_n + b_n Tx_n + c_n u_n,\ x_1 \in X,\ n \geq 1,$$

and the modified Ishikawa iteration method with errors $I_{me}(T, x_n, a_n, b_n, c_n, a_{n'}n, b_{n'}, c_{n'}, u_n, v_n)$ given by

$$x_{n+1} = a_n x_n + b_n T[a_{n'} x_n + b_{n'} Tx_n + c_{n'} v_n] + c_n u_n,\ x_1 \in X,\ n \geq 1,$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$, $\{a_{n'}\}_{n=1}^{\infty}$, $\{b_{n'}\}_{n=1}^{\infty}$, $\{c_{n'}\}_{n=1}^{\infty}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = a_{n'} + b_{n'} + c_{n'} = 1$, and which satisfy other suitable conditions; $\{u_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are bounded sequences in X. The relationship between the iteration schemes of Liu and Xu are now well known (see for example [21,93] for comparison of the two schemes).

With the introduction of these iteration methods with errors, many authors started studying the stability and the almost stability of the schemes (the iteration methods with errors) with respect to certain operators especially the strong pseudocontractions and ϕ-pseudocontractions (see for example [2,53-54,138]).

Recently, some authors (see for example [56,67,111-113,128]) studied the *three-step iteration* *method* $T_s(T, x_n, y_n, z_n, \alpha_n, \beta_n, \gamma_n)$ given for arbitrary $x_1 \in X$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T[(1-\gamma_n)x_n + \gamma_n Tx_n], n \geq 1,$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying certain suitable conditions. Strong convergence of this iteration method to fixed points of certain classes of operators has been proved by some authors (see for example [56,67,111-113,128]). Operators already studied include the strong pseudocontractions and the ϕ-strong pseudocontractions.

We have made far-reaching contributions in the study of stability of iteration methods. In one of our latest contributions: Osilike, The Ishikawa and Mann iteration methods (with errors), stability, and three-step iteration methods. Fixed point theory and applications. Vol. 7, 135-145, Nova Sci. Publ., New York, 2007, (MR2355761 (2008h:47127)), we established among other things, certain intimate relationships between the stability of the original Mann and Ishikawa iteration methods with respect to a mapping T, and the strong convergence of the Mann and Ishikawa iteration methods with errors to a fixed point of T. Our results indicated that it is unnecessary to study the Mann and Ishikawa iteration methods with errors for mappings for which the original Mann and Ishikawa iteration methods have been shown to be stable; or to study the stability of the original Mann and Ishikawa iteration methods for operators T for which the Mann and Ishikawa iteration methods with errors have been shown to converge strongly to a fixed point of T. Furthermore, our results indicated that the study of the stability of the iteration methods with errors may be unnecessary. We also showed that the convergence of the three-step iteration method studied recently by some authors for certain classes of mappings can be obtained as simple corollaries of the corresponding convergence theorems for the Mann and Ishikawa iteration methods with errors.

Other major contributions which we have made in the area of stability of iteration methods and iteration methods with errors include the ones that appeared in the following articles:

- Osilike, *Indian J. Pure Appl. Math.* **26 (10)** (1995), 937-945, (MR1364086 (96i:47104)).
- Osilike, *J. Nigerian Math. Soc.* **14/15** (1995/96), 17-29, (MR1775011 (2001d:47091)).
- Osilike, *Indian J. Pure Appl. Math.* **27 (1)** (1996), 25-34, (MR1374885 (96m:47104)).
- Osilike, *J. Math. Anal. Appl.* **204 (3)** (1996), 677-692, (MR1422766 (97m:47096)).
- Osilike, *J. Math. Anal. Appl.* **213 (1)** (1997), 91-105, (MR1469362 (98g:47054)).
- Osilike, *Indian J. Pure Appl. Math.* **28 (8)** (1997), 1017-1029, (MR1470119 (98j:47132)).
- Osilike, Indian J. Pure Appl. Math. 28 (1997), no. 9, 1251-1265, (MR1605502 (98m:47090)).

- Osilike, *J. Math. Anal. Appl.* **227 (2)** (1998), 319-334, (MR1654162 (99h:47071)).
- Osilike and Udomene, *Indian J. Pure Appl. Math.* **30 (12)** (1999), 1229-1234, (MR1729212 (2000m:47081)).
- Osilike, *J. Math. Anal. Appl.* **250 (2)** (2000), 726-730, (MR1786094 (2001h:47096)).

5 Concluding Remarks and Acknowledgements

We have tried to summarize what we have been doing in the area of fixed point theory and applications from our very unfavourable environment. We are still working and making contributions in this area despite the fact that the conditions appear to be getting more and more hostile to research in sciences. In the area of postgraduate supervision, I have been making my own modest contributions. I started early as "an apprentice supervisor" under my academic mentor-Professor C.E. Chidume, FAS. In 1999 two students I guided successfully completed their M.Sc. degrees in Functional Analysis. One of them was S.C. Aniagbaosor of blessed memory, and the other person is Dr. D.I. Igbokwe who I later guided to a successful completion of his Ph.D. in Functional Analysis in 2002. Dr. Igbokwe won the first prize of the Vice-Chancellor's Postgraduate Prize for the 2001/2002 Session, and he is now an Associate Professor (Reader) at University of Uyo. In 2002, Dr. A. Udomene also completed his Ph.D. under the supervision of Professor Chidume and myself. Dr. Udomene lectured briefly at University of Porthacourt before he left for University of KwaZulu-Natal, South Africa. Other successful postgraduate supervision are:

F.O. Isiogugu	M.Sc. (2006)
B.G. Akuchu	Ph.D. (2006)
Y. Shehu	M.Sc. (2008)
P.U. Nwokoro	M.Sc. (2009)
O. Dokubo	M.Sc. (2010)
E.E. Epueke	M.Sc. (2010)

Dr. B.G. Akuchu also won the first prize of the Vice-Chancellor's Postgraduate Prize for the 2006/2007 Session.

We have many other postgraduate students who are struggling with the very bad academic environment we have in our country. We are exploring new ground in our area of research, and we are also exploring opportunities beyond our immediate environment. One of my Ph.D. students—Mrs. F.O. Isiogugu had just completed her third visit to University of KwaZulu-Natal, Pietermaritzburg, South Africa. She visited with the Sandwich Postgraduate Fellowship of the Organization for Women in Science for the Developing World (OWSDW) [formerly the Third World Organization for Women in Science (TWOWS)].

We have had opportunities to work at other parts of the world and we are not in doubt that our environment here is very unfavourable for any meaningful research work in basic sciences. I agree completely with very many Nigerians that described the Nigerian environment as hostile to research activities. Our government has not adequately supported research at high level in basic sciences and the level we are in science and technological development clearly portrays this. No country can advance in science and technology without adequately supporting high level research in basic sciences. Our communication facilities are still extremely inadequate and the available ones are very expensive when compared with the meagre remuneration we receive. We spend most of our time in search of basic amenities. Power supply is very limited and security of life and property which is one of the major functions of government is also seriousely threatened and insecurity is now a major threat to our socio-economic development. Good health facilities are not in existence and the inadequate ones that exist are not affordable. Our remuneration is still very poor and "poverty anywhere is a threat everywhere" (ILO Declaration (1944)). Few individuals who by God's uncommon grace were able to get themselves out of the "closed doors" have demonstrated that we can do well in science but collectively we are nowhere. We had Mathematical Reviews by American Mathematical Society (AMS) in our Library up to December 1989 issue 89m. It is no longer available! MathSciNet which consists of Mathematical Reviews and Current Mathematical Publications is available on the internet but only accessible on subscription. Our great University is yet to subscribe and such an important facility is elusive to us.

Mathematicians are fast becoming "endangered specie" Young people are not encouraged by the government to study the subject and the few people in the area are overstretched (or they overstretch themselves?). In the whole of Eastern Nigeria, there are not enough quality academics to deliver the services expected from the department of Mathematics, University of Nigeria alone, yet we have more than ten (10) tertiary institutions that offer various degrees in Mathematics in this region. Some of the Universities in the Country do not have even a single permanent staff in their Mathematics Department. One is still looking forward to the time when young graduates of Mathematics should be given authomatic employment and attractive remuneration in federal and state civil service, and when graduates who choose to pursue their postgraduate studies in Mathematics should be given full scholarship and support by the government. I do not even see any reason why undergraduates of Mathematics Departments should not be under full government scholarship.

The National Mathematical Centre which Professor J.O.C. Ezeilo and his colleagues laboured to established to develop high level research in Mathematical Sciences appears no longer to be pursuing the objectives of these heros. These heros discovered long ago that high level research in Mathematical Science is not only invaluable but indispensable for national development of any nation so they made tremendous effort to establish the Centre to run just like the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. It started well and many benefited from it and it should have been a rallying point for Mathematical Scientists in our country and beyond to complement facilities like current literature, MathSciNet, etc that are lacking in our institutions. Unfortunately however, it is suffering from the usual "disease" (you know better) of our country and is yet to leave up to the desired expectation.

Acknowledgements. In all things I give glory to the Almighty God who by His infinite mercy, grace and loving kindness lifted me up from dust and ash heap just as the Psalmist asserted in Psalm 113 Verse 7. By His grace, I attended primary and secondary schools; by His grace, I studied at the only University of Nigeria as a "day student" from my brother's residence (then at Obollo road, Nsukka) and graduated with first class honours in Mathematics; by His grace, I operated from my brother's residence and obtained M.Sc. and Ph.D. in Mathematics from University of Nigeria; by His grace, I obtained the DICTP, Trieste, Italy; by His grace, He lead me through Nsukka-Enugu-Nsukka for three (3) years when I was teaching at the Institute of Management and Technology (IMT), Enugu; by His grace, I was found worthy to teach at the University of Nigeria; by His grace, He lifted me from Lecturer II at first appointment at University of Nigeria in November 1, 1991 to full Professor of Mathematics on October 1, 1998, by His grace, He keeps me alive to this day to present this Inaugural Lecture! To Him be all honour, glory, praise, adoration and dominion in Jesus Name. I appreciate all whose spiritual lives have positively affected my life and my entire family.

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of Nigeria have made it possible for me to continue. Ever since he taught me "Metric Space Topology" and other courses in third year, he has continued without bound to encourage me. He supervised my M.Sc. Project Report and after registering for Ph.D. with him in 1988, he tried to get me a job at University of Nigeria and when it was not forthcoming, he took me to the Head of Department of Systems Science, IMT, Enugu-Dr. B.I. Eke and recommended me for a job and I was appointed Lecturer II. He made it possible for me to be at the Abdus International Centre for Theoretical Physics, Trieste, Italy for the DICTP in 1991/93 Session (He was the coordinator of the Programme). I completed my Ph.D. under his supervision in 1994 and he facilitated so many grants and research visits for me. Professor Chidume I thank you and your entire family. My sincere gratitude also goes to Professor Giovanni Viddossich of Scuola Internazionale Superiore di Studi Avanzanti/International School for Advanced Studies (SISSA), Trieste, Italy who taught me in the DICTP programme and supervised my DICTP Project Report. I appreciate Professor Emilia Mezzetti and Professor Bruno P. Zimmermann of University of Trieste who also taught me during my DICTP programme. Professor Giovanni Vidossich and Professor Emilia Mezzetti wrote recommendation letters in support of my first and second applications for Regular Associate of ICTP. Professor Simeon Reich of Technion-Isreal Institute of Technology, Professor B.E. Roades of Indiana University, Bloomington, USA and Professor C.H. Morales of University of Alabama in Huntsville, USA at one time or the other communicated with me, sent reprints and gave valuable advice and suggestions. I am grateful to them and all other colleagues both in University of Nigeria and beyond who had associated with me and contributed in one way or the other in my career. I passed through great teachers at University of Nigeria, Nsukka-Professor J.O.C. Ezeilo, FAS, CON, Professor G.C. Chukwuma, Dr. A.D. Nwosu, Dr. A.C. Anyanwu, IR. E. Ukeje, Dr. C.N. Nwoke, Dr. I.S. Adjero (all of blessed memory); and Professor J.C. Amazigo, FAS, Professor S.C. Rastogi, Dr. E.C. Obi and Dr. A.K. Misra. I salute all of you. Professor J.C. Amazigo's way of impacting his knowledge had remained memorable in my mind and his tremendous exploits in the field of Mathematics, his leadership capabilities, his humility and personality will inspire any person to pursue a career in Mathematics and it indeed inspired me. I appreciate the kind cooperation I have been receiving from all my colleagues in my Faculty and Department. My special appreciation goes to Professor M.O. Oyesanya for the wonderful citation. I salute all the teachers I passed through during my Primary and Secondary School Education.

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No. UN/SRG/94/39 worth twenty-eight thousand (N28,000.00) Naira only which it awarded me in 1997. I also thank the University for several permissions to travel out of the country for research visits which it granted me "at no cost to the University".

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I must say that I lack adequate words to acknowledge the assistance, motivation, inspiration and encouragement I have received from the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. Since 1992/93 Session when I pursued the DICTP there with full Scholarship, it has continued to be the pivot (the driving force) of my research activities. Apart from several short visits, conferences, workshops and schools which I have attended with full sponsorship by ICTP, it has awarded me Regular Associate twice. Since 2003, I have been accessing several e-journals which include all Elsevier and American Mathematical Society (AMS) Mathematics Journals under the e-journal programme of the Centre for developing countries. The ICTP Library has on several occasions supplied me on request, articles needed for research and which I was unable to source locally. The Abdus Salam ICTP, your contributions in the growth of my career is invaluable, I thank you immensely.

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