DELAY AND CONTROL IN DIFFERENTIAL EQUATIONS: APOGEE OF DEVELOPMENT
BY

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PREAMBLE

Whenever I embark on teaching any course, however abstruse or elementary, I am first of all used to trying to explain the major concepts involved from their most fundamental notions. This enables the student (gifted, average or even below average) to be in a position at the very beginning to follow what the knowledge to be imparted is all about.

An inaugural lecture is an address which a long serving and an experienced University teacher usually delivers mostly to the academic community as a whole as an indication of his arrival at the much desired post of **Professor**. It is his first lecture which, as it were, serves as an x-ray of what the research he had been conducting so far had been throughout all the long years of his difficult struggles from the first rung of the academic ladder (as assistant Lecturer or Lecturer II as the case may be) to its summit **(Professor**). The audience is usually made up of both the academic and the non-academic staff of the institution as well as students and others from within and outside the ivory tower.

Although the inaugural lecture is in theory supposed to be the initial address of the academic to announce his ultimate entry into the jealously guarded bracket of the most senior University teachers in his University; in practice, however, it marks the beginning of the end of the Professor's career. This is so because the lecture should in reality be given after the professor must have stayed long enough for him to have conducted sufficient research to present at the inaugural lecture. This is more so in the great University of Nigeria where until very recently the processes and the protocols involved in promotion are known to be delayed, and delayed, and (in fact) so delayed that many recipients in the last decade were dead by the time their promotions were published. This, of course, is not the type of ***delay*** I will be discussing in my lecture shortly. **Delay,** especially in differential equations, constitutes a major part of my research work and so I now tell you with some measure of authority that in general delay as a rule is good and indeed desirable. But as it is usually said – "*Every rule has an exception*" Delay in promotions at the University of Nigeria is an exception to this Rule.

Main Difficulty of Inaugural Lecture

Mr. Chairman, the Vice-chancellor, Sir, ladies and gentlemen, the inaugural lecture when compared with the routine undergraduate or postgraduate lectures poses a serious problem whose solution is by no means trivial. In the latter case, each course has a well-set-out syllabus. Besides, each member of the class must have successfully passed through a lower pre-requisite course. On the other hand, in the inaugural lecture the audience are not so drilled to follow the lecture. Besides, there is no well defined lecture content which the people so innocently assembled could have consulted earlier on so as to be familiar even a little bit with the subject of the lecture. This is the main difficulty of the inaugural lecture.

In the circumstances, Mr. Chairman, and respected members of the audience, the job at hand must still have to be done. At times I will get down to such basic rudiments of the subject that everybody in attendance will be able to make some meaning out of the lecture. At other times, however, I will move up so relatively high that the academics especially the high-ups both seated here and others elsewhere will not begin to wonder what type of research work must have led to this particular inaugural lecture. I, therefore, hereby apologize to each member of this august assembly for his temporary disappointment as we move from stage to stage in this lecture in which the difficult task I have set for myself is to carry everybody along.

Well, as I pointed out earlier, it turns out that inaugural lectures are predicated mostly on the research the professor must have carried out preferably over a long period of time.

RESEARCH

Research is the act of inquiring deep into some phenomenon with the view to unraveling the ***"mystery***" clouding it. Usually we carry out research about what is currently unknown; research seeks to dig beyond the obvious. However, among the many possible explanations of the word "research" the most interesting to me, perhaps, is the one given satirically by the then Dean of Student Affairs in the University. During the orientation of freshmen in the 1966/67 academic session, he defined research as follows; it is:

"**studying more and more about less and less until one knows almost everything about nothing"!**

The Dean in question was Dr. E. N. Ukpaby, B.Sc.(Tuskegee). M.A.(Atlanta) , Ph.D.(Bradley).

This definition by Dr. Ukpaby, I believe, is for entertainment purposes by that philosopher. In the course of this lecture I will have the opportunity to bring to the knowledge of this audience a more comprehensive and even more satirical definition given, I believe, to drive home some serious ingredients of research.

0. GENESIS. Mr. Vice-Chancellor, Sir, distinguished ladies and gentlemen, it is my wish to discuss some six theatres of learning which readily guided my journey from my earliest days till now. By even mentioning them and some of the **“dramatis personae”** in such theatres who either as teachers or colleagues (class mates or school mates in general) have helped by design or indirectly or by mere chance influenced my development in one way or the other. I derive some measure of pleasure as this is, perhaps, the only time I have the opportunity of acknowledging their contributions in my somewhat gradual and difficult journey to self-development. They hereby appear in some chronological sequence of time.

0.1 St. Mary’s (now Township) School, Ogrute, Enugu-Ezike. This is where I had my Primary Education. It can be regarded as (and it is) the first term of an Arithmetic Progression which this lecture will expose. When I started schooling, there were no nursery or kindergarten schools for pre-primary preparatory education. The period of education was then eight years. A class called **Prima** intervened between the initial class called **A-B-C** and Primary One which gradually progressed to Primary Six. As soon as my set scaled through Prima it was abolished! Soon after that, the A-B-C was similarly abolished and Primary School period since then became six years.. This meant that I had a **delay** of two years in the Primary School.

My Primary School teachers included Mr. F. O. Thompson, Mr. C. C. Nebo (a renowned Head Master). Mr. Gabe. Attamah, Mr. Francis Mbaeze, and Mr. Emmanuel Idoko. Many of them are still alive. Some of my classmates were Mr. Ignatius Ogbu, Linus Mbah (UNN retiree), Mr. Patrick Agashi (died as a Ph.D. student being supervised by Dr. Iffih just before completing his program), Boniface Abugu, Nicholas Abugu, Benedict Ugwu, Paul Ogbodo, Sylvanus Abula, Nicholas Abugu, Fredrick Ali (NEPA retiree), Jonathan Abugu.

At that time, the seats for A-B-C and Prima were not moveable. They consisted of ridges of strong clay constructed in parallel rows. They were called **OKPO**. Every Friday the female students repainted OKPO with green, soft leaves. For reading at home, whenever kerosene became scarce we used to create "bush lamps" with broken earthen pots with red palm oil as gasoline. A long old cloth is coiled and rests in the palm oil leaving one end protruding at the edge of the pot. It is this end that is lighted. One funny thing about this type of local lamp is that at irregular intervals very hot oil dropped on the wooden table on which the lamp was placed, leaving permanent black crevices on the table. Within a single month a table may acquire as much as seventy such scars! One such table we used continued to lie in our compound till the Nigerian civil war made us lose it! I was used to looking at that table in those days from time to time. In spite of all odds, however, some of my colleagues and I were able to gain admission into Secondary Schools of our choice.

0.2 Christ the King College (C. K. C.), Onitsha. In our days, there were very few Secondary Schools around. St. Teresa's College was the only one in the Nsukka Senatorial Zone as a whole. In Eastern Nigeria, there were close to eight Catholic Secondary Schools for boys, namely, C. I. C., Enugu; S. P. C., Calabar; S. T. C., Nsukka; B. S. C., Orlu; C. K. C., Onitsha' Holy Ghost College, Owerri; and Stella Maris College, Port Harcourt. The entrance examinations for all these Colleges were always fixed the same day. One had to choose ab initio which one of them one wanted to attend. I was admitted to C. K. C.

The teachers in the College then were mainly foreigners mainly Irish Reverend gentlemen such as Rev. Fr. J. Fitz Patrick (Principal), Rev. Fr. Mc. Cabe, Rev. Fr. B. Russell, Rev. Fr. Smith, and Rev. Fr. Galt (from Trinidad). There were Indians such as Mr. Mathew (Biology), Mr. Mathani (Chemistry), and Mr. Ignatius (Physics). Two from apartheid South Africa were Mr. Ntshona (Science) and Mr. Ranque (Mathematics). Some Nigerian teachers were Mr. Onuoha, Mr. J. Onuorah, Mr. P. Eze-Okeke, Mr. R. Okafor, Mr. Iwobi, Mr. S.R.Okafor, Mr. R. C. Okagbue, Mr. L. Eneh. In my final year in 1963 Rev.

Fr. N. C. Tagbo took over as the first African Principal from the departing Irish priests.

In order to prevent her good students from going to other Institutions for their Higher School Course, the College formed the habit of offering direct entry(admission into the Higher School Course without entrance examinations) to her good students. I was one of the eight so selected in my set. By then I was already encountering much financial problems. Consequently, I went and explained that to Rev. Fr. Tagbo. He readily allowed me to read for the two years without paying anything provided I would settle the bills after finishing the course. When I finished the course in December 1965, I got my first teaching appointment at Q. R. S. S., Nsukka starting in January 1966. From my salary of twenty one pounds (that is, forty two Naira) per month I made a spirited attempt to pay up my indebtedness to the college but could not clear it. It remained forty seven pounds and ten shillings (that is, some N94.50) by the time I needed to enter the University in September 1966.

I had applied for and gained admission into two Nigerian Universities. One was the University of Nigeria which offered me a three-year program in **Mathematics**. The other was the University of Lagos which offered me admission to study **Mechanical Engineering** on a three-year program. Then Nsukka was offering four years in Mechanical Engineering and that was why I did not apply to read Engineering at U.N.N. To me then time was of great essence on account of what I narrated about my C. K. C. experience in the paragraph above. That was in 1966 when Nigeria was plunged into deep, bloody, patricidal crisis. About the time we were to enter the University then Colonel Chukwuemeka Odumegwu Ojukwu (Military Governor of Eastern Nigeria) announced that anybody planning to leave the region then was doing so at his own risk. That announcement immediately settled my choice of course to Mathematics and U.N.N..

When I was invited for interview (about May 1966) for the Eastern Nigeria scholarship, I traveled to Onitsha to see if I would be allowed to collect my certificates. Their release was not certain since I was still owing the College then. Fortunately for me both Rev. Fr. Tagbo (the Principal), and the College Secretary – Mr. Patrick from Asaba –were present in the College by the time I arrived at Onitsha. The Principal asked me what I came for. I did not reply. He asked again (and of, course, he knew the answer himself since many people were streaming in daily to the College for the same purpose – collection of their certificates for the scholarship interview}. As I did not answer his questions, Rev. Fr. Tagbo smiled lovingly and thereafter ordered Patrick to release my certificates to me unconditionally! As I reached the College gate (full of joy) the Principal called me back. Behold! He had manually written a powerful little note to the Scholarship Board, urging them to ensure I was awarded the scholarship as the certificates were not enough to interpret my need for the scholarship award.

When I went for the interview I slipped the piece of paper to the Chairman and thereafter he informed the other members of the Scholarship Board that my interview was no longer necessary. I was awarded the scholarship just like that! Chris Agunwamba of Statistics Department was for that interview too. He was awarded the scholarship to study combined Mathematics/Physics. Together with the serially charred table at Ogrute, that Rev. Fr. Tagbo's hand-written piece of paper remains about my greatest loss during the civil war.

It was as late as December 2003 that my family – my elder brother Paddy Eke and my wife Mercy of the Personnel Unit – with the help of my bosom friend Sir C. Ewesiobi (Comedy) sought for and located Rev. Fr. Tagbo. He had since retired from service and was attached to St. John's Catholic Parish at Odoakpu in Onitsha municipality. The long delay in trying to meet Rev. Fr. Tagbo so as so settle my indebtedness to C. K. C. under his leadership was caused by my waiting for what I called "a more suitable time". For many years my wife piled pressure upon pressure on me until I succumbed in 2003. We missed Rev. Fr. Tagbo when we arrived at Onitsha on December 31 2003..

Eventually we located him in his home town Awkuzu where he was relaxing with the other members of his family for it was Christmas period. The issue of my indebtedness was ended that day. I am very grateful to that man of God.

I spent seven years in C. K. C. Many of my classmates or associates whose names easily come to mind include: Mike Enenmuoh, Anthony Adizua, Peter Chukwumah, Gordian Nnabuenyi, Emma. Nwadiogbu, Greg. Nwoye, Peter Obuasi, Aloy Okonkwo, Emma. Odimgbe, Greg. Oraekie, Mike Ajegbo (former Senator and Proprietor of Minaj TV International), Frank Okonkwo, Christian Onwuzo, Charles Ude, Isaiah Okoye, Alex. Okeke (Aco-Ecology), Richard Ugwu, F. Okoli (former Deam of Social Sciences), C.E.Okeke (former Dean of Physical Sciences), Fred. Ugwuaku (a former Commissioner, old Anambra State), Francis Blain (mulatto from the Gambia), George Jobarte (the Gambia), F. Igboama, C. Igboka, J. Ifem, J. Ufoh, M. E. Osisiogu, L. Ogoukwu, Emma. Nnebocha, and so on. I have not met most of these colleagues since 1966 and I am aware some of them are dead.

Our study prefect in my first year was P. Ngoddy (UNN professor) who was then in the Lower Sixth Form of the Higher School. I close this section by stating emphatically that Rev. Fr. Tagbo's immeasurable good gesture to me remains (and will continue to remain) my greatest source of inestimable joy then and thereafter. It helped me enter the University of Nigeria without difficulty in September 1966.

0.3 The University of Nigeria, Nsukka I arrived at the University of Nigeria, Nsukka (122 Balewa Hall) for a three-year degree course in Mathematics in September 1966. The class remained fluid till the third year. The reason was that students could change (and many did change) their program even on arrival at the Campus.

The way courses were arranged in those days was very dangerous. Each of our Departmental courses carried 12 units. In our final year, pressure on the Departmental Authorities forced them to reluctantly create just two 6-units courses. The Social Science course G S 103) carried 15 units so that its lectures and seminars were held six times a week (that is, every day except Sundays). The implication of those heavy credit loads was that the maximum number of courses which a student could fail and hoped to continue his studies was ONE! Anyone who was referred in two courses simply failed out of the Department. At that time, the Department was called the D**epartment of Mathematics, Statistics and Astronomy.**

As a result of the prevailing circumstances then, only three of us reached the final year on schedule as Mathematics majors. Fortunately, the three of us graduated in June 1972 instead of June 1969 on account of the Nigerian civil war an undesirable **delay** of. Three years! The three of us were Israel Mbakwe (probably a retired Secondary School Principal by now), Prof. S. O. Oko (twice DVC at Ebonyi State University), and myself.

We had many eminent lecturers and Professors such as P. C. Chaudhuri (First Head of Department), J. O. C. Ezeilo (who later became Vice-Chancellor), P. J. van Albada, and G. C. Chukwumah, J. N. Adichie. The Lecturers included O. K. Mitter, I. D. Seth (a strong Indian Applied Mathematician who on account of the civil war relocated to the University of Lagos where he later died), G. D. Dikshit, E. N. Chukwu. (later the founding Vice-Cnancellor of the Federal University of Technology, Yola and my first Ph.D. supervisor), E. Ukeje, Greg. Emembolu, and A. D. Nwosu. My seminar Leader in Social Science was G. Atuanya and that for the Use of English was Pol. Ndu. One thing deserves mention here. Emeritus Professor Ezeilo's method of teaching remains a marvel to many of us especially those of us who were fortunate to have passed through him again and again. When he was in class everybody always felt he has understood everything as Ezeilo could and always did explain even the most difficult concept very easily with minimum effort. After some hours, however, the thing explained would appear like nightmare again! A man of great patience, Ezeilo would repeat the explanation next time

he came to class if the students so desired and requested, notwithstanding that the concept so explained again and again may not be necessarily better comprehended even this time around!
After graduation, I left for the then Bendel State where I taught for about a year before coming back to the University for a Master's degree program.

0.3(1) U. N. N. I came back to UNN in October 1973. We were two in the Program – I and Professor C. E. Chidume who later (1985 – 1990) with Professor J. O. C. Ezeilo jointly supervised my doctorate degree project. Although as a Junior Fellow Chidume was scheduled to do a straight Ph.D. degree, after our course work in 1974 he later left for the U.S. to study Operator Theory instead of Differential Equations mounted in our Department then. He later came back and joined the staff of the Department. Long before then, the Department had been split into two separate Departments, namely, Mathematics and Statistics. Then Astronomy was merged with Physics. Our lecturers in that program were Prof. Ezeilo, Dr. E. C. Obi, and Dr. D. U. Anyanwu who was also my supervisor. I was awarded the M.Phil. degree in the **Asymptotic Theory** of Differential Equations in September 1976.

0.3(2). U. N. N. Again. I started. a Ph.D. degree program in Differential Equations in 1977 again with Dr.. D. U. Anyanwu again as supervisor. But at a

Conference in C.E.C here at UNN in 1978 organized in honour of Prof. Ezeilo by Mathematicians from various Nigerian Universities, I met Prof. E. N, Chukwu (that time a Reader at the University of Jos). Besides, Prof. Ezeilo was leaving Nsukka to become the Vice-Chancellor of the Bayero University, Kano. I, therefore, decided to diversify my academic base. as Chidume did earlier. So I informally withdrew from the course at Nsukka and left for Jos initially on part-time but later in 1981 started to pursue a full-time Ph.D degree course in **Control Theory.**

When I was still in Jos doing some new course work, Prof. Chukwu was appointed the pioneer Vice-chancellor of the new Federal University of Technology, Yola.. On completing my course work in Jos I followed Chukwu to Yola. Soon after that, however, he lost the job and left for the U.S. However, prior to that he had arranged for me to travel to the U.S. (the University of South Florida, Tampa, Florida) for some research work under Professor A. G. Kartsatos. By July 1985, I returned to Nigeria. Since Chukwu was no longer around I got a fresh admission at U.N.N. which culminated in my earning the Ph.D. degree in April 1990 in **Nonlinear Functional Differential Equations**, an amalgam, as it were, of Differential Equations and Functional Analysis jointly supervised by Prof. Ezeilo (renowned scholar in Differential Equations) and (then Dr.) Chidume-expert in Functional Analysis.. The work I did in Florida constituted the major part of my Ph.D. project.

As stated earlier, in a bid to change my research base I decided to go to Jos.

0.4. The University of Jos. In October 1981 (having procured study leave with pay from the U.N.N), I left Nsukka for the University of Jos. There I registered for three new courses – Functional Analysis, Functional Differential Equations, and Control Theory. At Jos I was taught by Prof. (then Dr.) O. Akinyele (Ibadan), Prof. E. N. Chukwu, and Dr. J. Wilkowski. We finished the lectures in all the courses and took the examinations on schedule. It was about that time that Prof. Chukwu was appointed the Vice-Chancellor in Yola but was later removed. He migrated to the U.S. (where he still lives) and I later went to Florida.

0.5. The University of South Florida, Tampa. I left for Tampa on 2nd January 1985. I was appointed a Teaching Assistant and had to take Real Analysis which was taught by Prof. A. Mukherjea. I worked with Prof. A. G. Kartsatos and Prof. M. E. Parrott. I completed the second semester and the Summer session and returned to U.N.N. in July 1985 after much traveling between Nsukka, Jos, Yola and the U.S.

0.6. Research Institutes. There are other Institutions which helped to strengthen my research base. They are

Research Institutes. They are purely research-oriented Postgraduate Centres which by themselves may not award any degrees. There are three of such centres, namely, National Mathematical Centre (N. M. C.) Abuja-Nigeria; the Hanoi Institute of Mathematics (H.I.M.), Hanoi-Socialist Republic of Vietnam; and the Academy of Mathematics and System Sciences (A.M.S.S.), Beijing-Peoples Republic of China.

0.6(1) National Mathematical Centre, Abuja–Nigeria. The National Mathematical Centre (N.M.C)., Abuja was established through the initiative of Prof. J. O. C. Ezeilo who was also the foundation Director. Professor Ezeilo devised a journal-exchange arrangement whereby he annually bought several copies of the Journal of the Nigerian Mathematical Society and exchanged them with differential journals from various parts of the world.

Among other programs, the NMC mounts regular Foundation Programs, Workshops. And Seminars. It even established a Research Grant program. It has provision also for sponsored visits to the Centre for private individual research works. I benefited immensely from most of the NMC's programs. In 1992 I was a Resource Lecturer in a Postgraduate program organized by Prof. M. A. Ibiejugba of the University of Ilorin. In 1999 and 2001 I was the Organizer and Coordinator respectively of two Postgraduate programs at the Centre. Apart from

sponsored private visits to the NMC, I benefited from the first Research Grant of the Centre which enabled me to do some work on **stabilizability** [14].
The people who managed the NMC at the various times I was there were Prof. J. O. C. Ezeilo (Director); Prof. R. F. A. Abiodun (Deputy Director and Coordinator of Mathematics Programs and also Deputy Director at times); Prof. S. O. Iyahen (Deputy Director, coordinator of Mathematics Programs and at times Acting Director); Prof. A. O. E. Animalu (Director); and Dr. J. Daniel (Librarian).

The National Mathematical Centre is modeled after the Abdus Salam International Centre for Theoretical Physics (I. C. T. P.) in Trieste, Italy. I have visited I.C.T.P. three times (1986, 1988, and 2001). My first visit in 1986 was through the help of Prof. Ezeilo. He nominated Dr. C. Nwoke and myself for that. An arm of the ICTP – the Third World Academy of Sciences (TWAS) has consistently sponsored my visits to the other two Ternary Institutions. TWAS usually provides the air-tickets and some subsistence allowance while the host Institutions take care of the maintenance of the Visiting Associate under the framework of TWAS Associateship Scheme at Centres of Excellence in the South.

0.6(2). Institute of Mathematics, Hanoi – Vietnam.
The Institute of Mathematics in Hanoi, Vietnam is one of the Centres of Excellence in the south. Under the

TWAS Associateship Scheme of the ICTP I have visited the Institute four times (1997, 1999, 2005 and 2007). Each time, not only did I work in the nice Library of the Institute, the entire staff, from the Director down to the support staff showered me with unbeatable Vietnamese hospitality. The people I met in the Institute and who helped me immensely include Prof. Dinh the Luc, Prof. Ha Huy Khoai (former Directors), Prof. Vu Ngoc Phat (my collaborating research staff), Prof. Le Tuan Hoa, Prof. N. K. Son (wonderful expert in Control Theory, now in the upper echelon of the Vietnamese Government).Prof. N.V. Trung (current Director) Prof. Nguyen Viet Dung (current Deputy Director),Dr. Nguyen Xuan Tan, Mr. Chau, Ms. Ngoc (Guest House Manager), Ms K. P Thuy (Secretary to the Director), and Cao Ngoc Anh and her nice colleagues at the Bursary of the Hanoi Institute.

0.6(3). Academy of Mathematics and System Sciences (AMSS), Beijing – China. Under the TWAS Associateship Scheme I visited the Academy two times (2002 and 2003). The up-to-date Library and the active research team led and maintained by my host supervisor – Prof. Lansun Chen made each of my visits memorable both in terms of academic harvests and also the traditional hospitality of the people of China. During my first visit in 2002, I was fortunate to attend the International Congress of Mathematicians which was held in Beijing and was enabled not only the

Palaces of ancient Chinese Dynasties but also visited and climbed the famous **Great Wall** of China.

I am now closing this rather long but necessary part of my lecture and move on to the day's actual task.

GENERAL INTRODUCTION

The Chairman, Mr. vice-Chancellor, Sir, distinguished ladies and gentlemen, as I now move on to the topic of this lecture I wish to point out that my talk will be divided into three main sections, namely, **Elements of Differential Equations, Delay Differential Equations,** and **Control Theory.** There will be a fourth rather small concluding section which will highlight the **relationship between Mathematics (**of which Differential Equations is an important part) **and some other academic disciplines**. From time to time, in discussing Delay Equations and Controllability, I shall be highlighting my modest contributions in both areas and possibly those of some of my contemporaries and some other contributors to these fields whenever it becomes appropriate to do so.

In dealing with important concepts and notions in Mathematics (and I suspect it should be applicable to many or even all other disciplines as well) I will follow the directive of one of the leading Mathematicians in the United states of America who has taught in a number of leading American Universities, Professor Paul R. Halmos [16] which enjoins:

"The right way to read Mathematics is to read the definitions of the concepts and the statement of the theorems, … ."

This is what I will be doing as I discuss each of the concepts of this lecture.

1. DIFFERENTIAL EQUATIONS.

1.1 Basic Notions. Any expression of equality (**=**) is called an **equation.** A differential equation is any equation which contains either **derivatives** or **differentials.**

Let us take some physical quantity initially of magnitude **y**, say. If **y** grows (or increases) after some time by an amount **dy**, its mew magnitude is then $y + dy$ Consequently, the difference between the final and the initial magnitude is called a **differential** and it is denoted by dy. In the same way, if x is another quantity different from y, its differential is given by dx. Hence, generally independent changes in various designated quantities y, x, s, t, … are given by their respective differentials, thus

$$dy, dx, ds, dt. \ \ldots \qquad \ldots 1.1(1)$$

However, not all quantities are completely independent (of other quantities). A common example is the **velocity** (or speed) of a moving object or particle. The velocity v of a moving object which covers a total distance s in time t is by commonplace definition given by $\frac{s}{t}$. More appropriately, to get the velocity at any given point within some interval we take it as

$$v = \frac{ds}{dt}$$

… 1.1(2)

Equation 1.1(2) shows the **change in s** corresponding to some **change in t.**

The quantity $\frac{ds}{dt}$ on the right hand side of 1.1(2) is called the **differential coefficient of s with respect to t** or the **derivative of s with respect to t**. The process of getting derivatives is called **differentiation** so that from 1.1(2) we say of its right hand side that **s** is differentiated with respect to **t**.

In many cases, a function which has been differentiated once can again be differentiated a second time. Suppose **a**, for instance, is the acceleration of a moving object. **a** is the rate at which the velocity **v** changes with respect to time **t.** That is

$$a = \frac{dv}{dt}$$

1.1(3)

Since from 1.1(2) v is given by $\frac{ds}{dt}$; it means that from 1.1(2) and 1.1(3) above we can write

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

1.1(4)

From 1.1(4) we then have $a = \frac{d}{dt}\left(\frac{ds}{dt}\right)$ and the right hand side of 1.1(4) is denoted by $a = \frac{d^2 s}{dt^2}$, that is

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

1.1(5)

and it is said to be the **second derivative of s with respect to t** The process of differentiation of derivatives can be continued (almost indefinitely) to get the third derivative, the fourth derivative, thus

$$\frac{dy}{dt}, \frac{d^2 y}{dt^2}, \frac{d^3 y}{dt^3}$$

1.1(6)

For other quantities y, x, t we can obtain

$$\frac{dy}{dt}, \frac{d^2 y}{dt^2}, \frac{d^3 y}{dt^3}; \cdots \quad \frac{dx}{dt}, \frac{d^2 x}{dt^2}, \frac{d^3 x}{dt^3}; \cdots$$

1.1(7)

With the above notations, we are now in a position to appreciate what a differential equation is. As we defined earlier, any equation containing differentials dy, dx or derivatives $\frac{dy}{dt}, \frac{dx}{dt}, or \frac{dy}{dx}$ is a differential equation. The following are examples of such equations

$$\frac{dy}{dx} = x \qquad 1.1(8)$$

$$x^2dx + xydy = 0 \qquad 1.1(9)$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0 \qquad 1.1(10)$$

We see from the above elementary developments that differential equations usually represent situations of physical quantities which are meaningful and useful when considering real life problems. Now, differentiation $\frac{dy}{dx}$ has an inverse called **integration** and it is denoted by $\int ydx$ indicating integration of y with respect to x. So, in theory, if we differentiate y with respect to x to get $\frac{dy}{dx}$, in general when we integrate $\frac{dy}{dx}$ written $\int \left(\frac{dy}{dx}\right)dx$ we expect to recover y.

The process of integration is used in solving differential equations. However, the solutions of most differential equations do not necessarily follow straightforward integrations. For instance, whereas the equation 1.1(8) can be solved by direct integration, neither 1.1(9) nor 1.1(10) can be so solved.

The method of solutions of differential equations is in itself a whole body of studies which does not concern us here. Besides, since differential equations usually represent real life situations which are physically meaningful, it is not always the case that such equations have solutions. The ones that concern us here are those that have solutions.

There are some spaces in which we describe differential equations and try to determine their solutions in those spaces for real life applications if possible. The easiest to understand and visualize to some extent is the so-called **Euclidean space.** The Euclidean space of dimension n (where n is some integral number) is denoted by E^n . .

1.2 The Euclidean space E^n

The Euclidean space is not hard to understand. Let us suppose that a body moves from the point **O** to another **B** a positive distance of units.

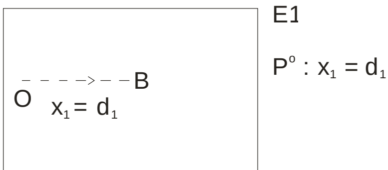


Fig 1.2 (1)

Assuming that there is no force which can divert the movement of the body or upwards (movement “in vacuo”), then in this case the body is said to have one-

dimensional motion in E^1 (1-dimensional space, say) (see Fig.1.2(1)
and written

$$x_1 = d_1$$

1.2(1)

On the other hand, if the displacement of a body from 0^1 to B^1 in a plane (such as the football field) can always be measured by x_1, x_2, along the mutually perpendicular directions respectively (see Fig1.2(2). The body is then said to have **two degrees of freedom**. The distance covered is d_2 with horizontal and vertical displacements x_1, x_2 respectively.

Using the well known Pythagoras Theorem it is clear that the displacement $O^1B^1 = d_2 = \sqrt{x_1^2 + x_2^2}$ as shown in the figure.

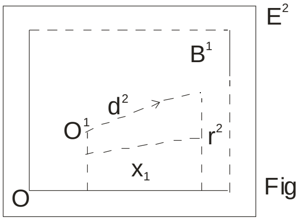
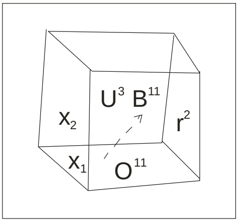


Fig 1.2 (2)

$P^1 : \sqrt{x_1^2 + x_2^2} = d_2$ ……………1.2 (2)

A space (as above) where a moving object has two degrees of freedom is said to be a 2-dimensional Euclidean space E^{21} .

It is now easy to deduce that in a space of 3 dimensions, the Euclidean space is denoted by E^3 and the distance d_3 moved in such a space can be calculated by applying the Pythagoras Theorem twice P^2 to get $d_3 = \sqrt{x_1^2 + x_2^2 + x_3^2}$



E^3

Fig 1.2 (3)

$P^2 : \sqrt{x_1^2 + X_2^2 + X_3^2} = d_3$ ………1.2 (3)

We can in theory deduce that in E^4 (the 4-dimensional Euclidean space) P^3 gives

$$P^3 = \sqrt{x_1^2 + x_2^2 + x_3^3 + x_4^2} = d_4 :$$

1.2(4)

But in the last case for E^4 we **cannot** draw the figure because in normal life we can visualize E^n only up to n = 3 (the 3-dimensional space).. However, we can even in an n-dimensional Euclidean space determine d_n after the application of the Pythagoras Theorem (n – 1) times to have for E^n

$$P^{n-1} = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_{n-1}^2 + x_n^2} = d_n$$

1.2(n)

So the Euclidean space is the most natural space where we can discuss solutions of differential equations readily. E^n is said to be of order n.

Apart from the Euclidean space, however, there are many other spaces in which meaningful analysis goes on even though we cannot draw the representative figures as was done in the case of Euclidean spaces such as the ones shown in Fig 1.2(1) to Fig.1.2 (3). But the most important thing about such spaces is that they exist and have far-reaching and definable characteristics which enable us to distinguish them from each other. Some of such spaces, if we take X as some non-empty set, are:

1. Metric Space (X, ρ)
2. Topological Space $(X, \Im)$
3. Hilbert Space (H)
4. Banach Space (B)
5. Sobolev Space $W_2^{(1)}(-,-)$
6. L_p - and l_p - spaces,
7. L_∞ - and l_∞ - spaces.

and so on.

Like many other researchers ([4], [24], [28], for instance) I have done some work in Banach Space [9]-[11]. These will be highlighted in the next section.

1.3. Existence of solutions of Differential Equations

Integration as the inverse of differentiation as we pointed out earlier is the major tool for finding solutions for differential equations. However, we are not discussing this point further. Rather we will like to introduce some alternative notations for derivatives and (ipso facto) for differential equations.

NOTATIONS. Let us recall the definition of the velocity **v** in 1.1(1), that is

$$v = \frac{ds}{dt}.$$

The formula shows that the distance **s** covered by the moving particle <u>depends</u> on t. We can thus say that s is a function of t and write

$$s = f(t) \qquad 1.3(1)$$

Consequently s is called the <u>dependent variable</u> and t the <u>independent variable</u>. In the same way we can have

$$y = f(x) \qquad 1.3(2)$$

At times y may have two independent variables x, t and in such a case

$$y = f(x, t) \qquad 1.3(3)$$

1.3(3) can give rise to a different form of differential equation. But in this lecture, we are not interested in variations of differential equations.

In the expression for derivatives $\frac{ds}{dt}, \frac{dy}{dx}$ if it is clear (beyond all doubts) what the independent variable is in each case, then we can write

$$s' = \frac{ds}{dt}, y' = \frac{dy}{dx}$$

1.3(4)

respectively. Alternative notations for these are $\dot{s}, \dot{y}$ Also, at times the notation **D** called the **differential operator** can be used for $\frac{d}{dt}, \frac{d}{dx}$ so that we now have

$$s' = \frac{ds}{dt} = Ds \quad \text{and} \quad y' = \frac{dy}{dx} = Dy$$

1.3(5)

These expressions (.)” or (:) and D(.) are extendable to higher order derivatives

$$s'' = \ddot{s} = \frac{d^2 s}{dt^2} = D^2 s \quad \text{and}$$

$$y'' = \ddot{y} = \frac{d^2 y}{dt^2} = D^2 y$$

respectively. In view of these, the differential equations 1.1(8) and 1.1(10) given earlier can now assume the following forms respectively

$$y' = x \quad \text{or} \quad \dot{y} = x \quad \text{or} \quad Dy = x$$

1.3(6)

$$y'' + y' - 2y = 0 \text{ or } \ddot{y} + \dot{y} - 2y = 0 \text{ or }$$

$$D^2y + Dy - 2y = 0 \qquad 1.3(7)$$

$$D^2y + Dy - 2y = 0$$

1.3(8)

Note that 1.3(8) can be written as

$$(D^2 + D - 2)y = 0$$

1.3(9)

We observe that only differential equations defined with derivatives (rather than with differentials) can be expressed in terms of primes (.), dots (.) or differential operators **D.**

Solutions

There are many types of differential equations. They are **ordinary differential equations** or **partial differential equations** according as the number of ensuing independent variables involved is one or greater than one. They are also classifiable as **linear** or **nonlinear** or according to **order** depending on the highest order of derivatives occurring in the equation. These ramifications do not concern us here. But the best and the easiest to handle are the linear ones of the first order (or those which can be reduced to that form). The most important thing is that we will for the purpose of this lecture operate with differential equations whose solutions exist and which exhibit such structures that are amenable to meaningful mathematical analysis.

Existence and uniqueness of solutions

The generally studied linear differential equations are differential equations in two-dimensional Euclidean space such as the x-y plane. Such differential equations are generally of the form

$$\left.\begin{aligned} \dot{x} &= f(t,x) \\ x(0) &= \overline{x}_0 \end{aligned}\right\}$$

1.3(10)

Here, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\overline{x}_0 = \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$ and differentiation is with respect to t. Conditions are prescribed for this equation and usually after a very long analysis, not only is it established that under the prevailing hypothesis that the differential equation 1.3(10) above has a solution **x(t**) but that such a solution is indeed **unique**. This literally means that if **x** is a solution of the equation, then if any other solution **y** exists, then **x = y**.

Comment The proof of the existence and uniqueness theorem for differential equations is not at all trivial. The mere statement of the theorem with the accompanying hypotheses is very long indeed. On top of this, the proof is so long and complicated that it should not concern us here. The most important thing is that we are assured of arriving at a correct result of establishing the existence of a unique solution in each case, a solution which can be harnessed for practical purposes that are of utmost importance in applications.

Application Very often when one completes a research work the question is usually put as to the immediate practical application of the result of the work. On this, Einstein the renowned physicist with interest in philosophy, maintains as follows [26]:

> **" The scientist finds his reward in … the joy of comprehension, and not in the possibilities to which any discovery of his may lead."**

Therefore, insisting on immediate practical applications of the results of research work is not necessary. So to continue to worry about the immediate applications of research conducted on l_p - spaces, for instance is not very necessary in all cases. One very important reason for this is that in the main, the sciences especially Mathematics and Physics are the loyal servants of Engineering. Most of the results obtained from scientific research can be (and are continually being) exploited by Engineers who produce utilities based on the scientific research results for the service of man. The most important thing is that scientists obtain valid results from carefully executed scientific research which usually expands the horizon of current knowledge, however small the expansion.

Consequently, if scientists do not give concrete examples or interpret the result of the research in terms of day-to-day application of the work, the research is in no way faulted. But if the scientist provides examples as they become available at times, then the better. The most important thing is that (in our case) differential equations have unique solutions (where applicable) which are of utmost importance in applications. We see how this is so especially in Delay Differential Equations.

2. DELAY IN DIFFERENTIAL EQUATIONS

In this section we shall discuss the structure of delay differential equations in both the Euclidean space and in Banach spaces. The usefulness of solutions to such differential equations will also be x-rayed in considerable detail.

2.1 Delay - notations and usefulness

To introduce the idea of delay as applicable to differential equations we will examine a simple differential equation having delay against the background of a similar one which has no delay associated with it.

Let us examine a simple linear ordinary differential equation of the first order given by

$$\left.\begin{aligned} \dot{x}(t) &= ax(t) \\ x(0) &= x_0 \end{aligned}\right\} \quad \ldots\ 2.1(1)$$

where **a** is a constant. Let us assume that 2.1(1) is satisfied by the function **x(t)** for all values of t in the relevant interval under consideration. If h is a given positive real number (h > 0), then the equation 2.1(1) will be said to have a delay **h** if instead of **x(t)** which appears on the right hand side of 2.1(1) we have **x(t – h)**. The new differential equation then becomes

$$\left.\begin{aligned} \dot{x}(t) &= ax(t-h) \\ x(0) &= f_0(t) \end{aligned}\right\}, \quad t \in [-h,0] \qquad 2.1(2)$$

The physical significance of x(t – h) is that the resulting equation contains the delay h and takes into account not only the present value of x(t) but also its past history x(t - h). Therefore, for an obvious reason, equation 2.1(2) is accordingly called a **Delay Differential Equation** or **(**alternatively**)** a **Functional Differential Equation.**

Delay Differential Equations are classified into two major groups, namely, those which have delay only in the state of the system such as 2.1(1) above specifically called **retarded functional differential equations** (R. F. D. E**.),** and those that have delay not only in the state but also in the derivative and are called **neutral functional differential equations** (N.F D.E.).

Simple versions of these systems are respectively given as follows:

$$\dot{x} = bx(t - h)$$
… (R.F.D.E.)

$$\dot{x}(t) - a\dot{x}(t - h) = cx(t - h)$$
… (N.F.D.E.)

where a, b, c, are suitable constants and h some positive number. The constant h is called the **delay** in the respective equations. For strategic reasons r > 0 is used instead of h.

For a functional differential equation another way of defining its delay component is as follows (Hale [15]).

$$x_t(s) = x(t + s) \qquad s\varepsilon[-h,0]$$
… 2.1(3)

that is, s is some non-positive number lying between - h and 0. Consequently, once one sees an equation of the form

$$\dot{x}(t) = f(t, x_t)$$
… 2.1(4)

in view of 2.1(3), one concludes it is a delay differential equation because of the presence of.

Under suitable conditions, the delay differential equation 2.1(4) has a solution. So also does the simple delay system 2/1(2). For an equation of the form $\overset{\times}{x}(t) = ax(t+k)$ for k positive, it is well known that it has **no solution** whatsoever as attested to by workers in delay equations ([3], [5], [15] among others). Since it contains a derivative it is a differential equation but it has **no solution**, Clearly, it is not a delay equation.

Applications Delay equations are much more useful than ordinary differential equations in many aspects of human development. Since they contain both the present as well as the past states x(t) and x(t – h) respectively of the system, this dual characteristic plays a pivotal role even in government business.

$(D^2 + D - 2)y = 0$ We can demonstrate this even with a local example. In the budgetary system of the government of the Federation of Nigeria, for example, the practice is that the President prepares the annual budget quite ahead of the time when it is supposed to be operational and presents it to the National Assembly for their consideration. The budget can become law only after its approval by the National Assembly and thereafter assented to by the President. If the National Assembly for one reason or the other starts arguing and arguing and fails to pass the budget on time for assent and implementation, there is a constitutional provision for the President to check what the budget for the previous year was x(t – 1), say, and starts using a designated percentage increase of the previous budget x(t – 1), say, and operate it meanwhile as if it is that of the current year x(t). Notice from my small analysis here that I have assumed the **delay** here to be h = 1 so that if **x(t)** is the **present year's budget** (which , perhaps, due to too much argument by legislators is not yet known), then a designated increase of that of the **previous year x(t – 1)** which is known is used. The cataclysm which would trail the non-release of money for the settlement of workers' salaries in time is better imagined than experienced. As it is with the President, so it is with all other employers of labour- Local Government Chairmen, Governors or Vice-chancellors.

Also, in aero-space engineering the instruments at the service of both the pilots and those in the control tower suggest the existence of some in-built mechanism with delay components which help to alter landing and take-off schedules as may be necessitated from time to time by the realities thrown up each rime. Without these, such very busy airports like the O'Hare International airport (Chicago), Kennedy International airport (New York), or the Heathrow Airport (London) will just be nothing but huge mass graves!

Similar arrangements obtain for the transmission lines in the railway systems [15]..

In fact, this same principle, I am sure, is applied even without explicit instruments by the members of the National Union of Road Transport Workers (N.U.R.T.W.) who control the movements of motor cars nationwide. The operators definitely do not know the principle behind what they do.

The traffic light in well organized cities are constructed with little delay mechanism. The red and blue lights interchange with little interval between the exchange of the colours so as to avoid collision of vehicles at the junctions.

Also in military science, the principle of delay is incorporated in all hardware for prosecuting modern warfare. This applies equally to both the instruments of offence and defence; otherwise energy, men and materials are simply wasted without results.

On the other hand it is well known [15] that in the predator-prey model which is often used in fish breeding the equations governing the birth rate of both the prey and the predators, the consumption rate of the predators are governed by well known **delay differential equations (see** [15], [30] and [31], for instance). If the fish population and other attendant variables are not monitored from time to time to ensure that they at all times conform to the dictates of the delay equations, then the population development of the fish will be impeded. Consequently, it will not be long before the pond becomes empty.

Medical science is not left out in the utilisation of delay. There is always some **time lag (**that is**, delay**) between the time the patient contacts some disease and the time the virus manifests in the body. Then and only then can a frontal attack by the medicine to be administered (all things being equal) exhibit maximum positive impact on the patient. So both the pharmacist who compounds the relevant medicine and the doctor or nurse who administers it should take note of any necessary **time lags** needed in the execution of their respective duties. The most common drugs for which the issue of delay is strictly obeyed is the puritone-chloroquine combined use to cure malaria. For those whose body constitution attracts itching when injected with chloroquine it is well known that chloroquine demands some **thirty minutes' delay** from puritone.

We can indeed continue to give more and more examples. The examples are indeed legion. With the above examples I have undoubtedly made sufficient case for **delay** as it occurs in my own chosen branch of scientific studies. I have demonstrated that under appropriate conditions delay is indeed an indispensable catalyst in human development.

I will now proceed to show the process of retrieving a solution from differential systems embodying delay.

2.2 Solutions of Delay Differential Equations

The beauty of delay differential equations is best appreciated when we examine the solution of about the simplest, linear first order delay differential equation in Euclidean space side-by-side with that of a corresponding equation without delay. The best systems to be so studied are 2.1(1) and 2.1(2); that is

$$\left.\begin{aligned} \dot{x}(t) &= ax(t) \\ x(0) &= x_{,,,,,}t \succ 0_0 \end{aligned}\right\}$$

… 2.1(1)

and

$$\left.\begin{aligned} \dot{x}(t) &= ax(t - h) \\ x(0) &= f_{00} \end{aligned}\right\}$$

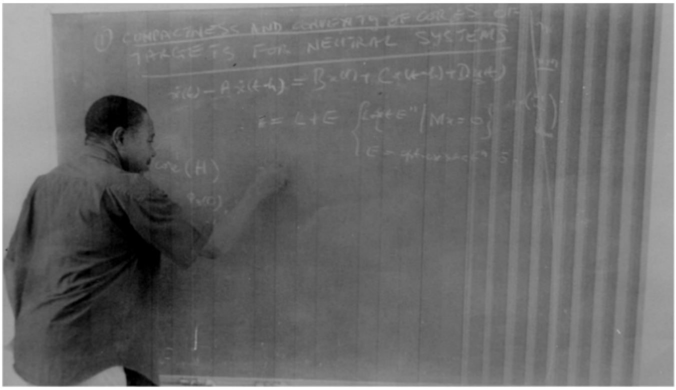
…. 2.1(2)

where in each case a is a constant and the systems are in E^{nj} 2.1(1) has no delay but 2.1(2) has delay h as shown. The solution of 2.1(1) without delay is very easy to obtain. For the equation 2.1(2) having delay the solution is not as easy to obtain.

In 1989 when I was trying to bring my studies for the Ph.D. degree to an end I picked the book of Driver [7] which gave a few steps towards the solution of 2.1(2). It was not at all clear. I then took it to one of my supervisors, Prof. (then Dr.) Chidume and complained that Driver’s partial solution was scanty. He then asked me to go and work out the solution up to where Driver stopped in detail, showing that I understood all the steps involved. Diver ended in three steps. Since my Ph.D. project was directly on Delay differential Equations, I had no alternative but to bend down on what my supervisor had ordered me to do. In the long run I was able to add a fourth step to what Driver did.

I have been talking about **steps** in the solution of delay equations because no such equation can be solved in a straightforward way, unlike the non-delay counterpart. It turns out that given an interval of time $I = [0, T]$, say, it is not possible to get in a single step a solution of any delay equation that is valid throughout the given interval. A popular method of solution which is in common usage is to break up the given interval I into many subintervals of equal lengths so that the resulting partition has n equal subintervals, say. Then we take the given equation and determine the solution which is valid in each subinterval starting from the least apart from the prescribed initial partition which has the initial function $\varphi(t)$ as a given solution. At the end of it all we match all the instalmental solutions. This process of solving delay differential equations is called the **method-of-steps**. It is excruciating. The difficulty of getting solutions to delay differential equations is understandable because they are very useful to man's development as we saw in the last subsection. Like all nice things, it is not surprising that they are difficult to procure.

Notwithstanding such difficulties posed by delay differential equations, however, many have worked in the realm of their study (see [2], [4] – [7], [27],



[15] [18] – [24], for instance). It is necessary at this point to briefly highlight some of my own contributions in delay differential equations..

2.3. Contributions to Delay Differential Equations

I encountered the subject in Jos where the duo of Prof. E. N. Chukwu and a Polish mathematician, Dr. J. Wilkowski, introduced me to it. Since it was new to me, I was made to register it along with Control Theory and Functional Analysis in the course work.

Preaching the **Gospel** of **Delay** in **HaNoi…**

At the end of the 1981/82 session in Jos I completed the usual examinations in the three courses. My knowledge in Delay Differential Equations was further strengthened at the University of South Florida, Tampa, Florida in the U.S. A. There, two professors – A. G. Kartsatos and Mary E. Parrott – drilled me in research work in the course, a subject in which both of them were then engaging in active research (see [18] – [24], and [27]).

My first work in delay differential systems [9] was on the existence and uniqueness result for delay differential equations of the type

$$\left.\begin{array}{l} \dfrac{du(t)}{dt} = A(t)u(t) + G(t, u_t),,, t \succ 0_t \\ u(t) = f(t),,,, t \in [-r, 0] \end{array}\right\} \cdots$$

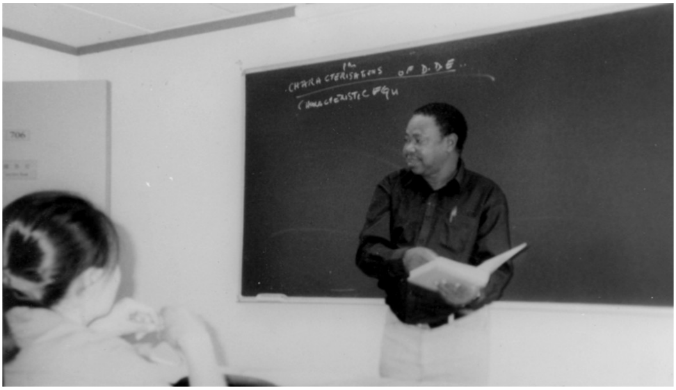
2.3(1)

in a special Banach Space X, say. In 2.3(1) A is a hyper-dissipative operator, G is a function which is uniformly continuous in t and Lipschitzian in u_t In accordance with the step-by-step procedure used in 2.1(2), the interval of solution [-r, T], T>0 say, was subdivided into n equal parts as solutions in the various subinterval were assumed as follows:

$$u_{nj}(t) = \begin{cases} f(t) \ldots t \in [-r, 0] \\ u_{n1} \ldots t \in [0, t_n] \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ u_{nj} \ldots t \in [t_{n,j-1}, T] \end{cases}$$

… 2.3(2)

The above assembly is called the **method-of-lines** for the delay differential equation 2.3(1). Under quite general conditions existence and uniqueness results for the given differential equation defined in the special Banach space X was established.



The next work [10] was based on delay differential equations of the type 2.3(1). In it, an important **convexity** result for the method-of-lines satisfying appropriate conditions was proved.

The last one we report here [11] is some delay differential equation of type 2.3(2) above. In that work, **alternative** to the method-of-lines was established, thereby proving that there exists more than one way of viewing the scheme 2.3(2) above for the purpose of analysis.

… and in **Beijing**

The most important thing about the above three works is that they all represent important solutions to various problems of delay differential equations defined in Banach Spaces. In the next section, where we discuss controllability, we will present a problem which combines delay with controllability [12] but this time in Euclidean space.

3. CONTROL IN DIFFERENTIAL EQUATIONS

In this part of the lecture, I wish to introduce some parameter which at times appears very innocently in some differential equations but when subjected to serious analysis is, perhaps, seen to be almost the alpha and the omega of the physical systems which such equations represent.

3.1 Basic notions of the control function u

Let us examine the dynamics of some particle (or body) initially situated at some point (x_0,y_0) of the x-y plane. Clearly, the space of operation here is E^2 ,the two-dimensional Euclidean space.

Suppose a certain system is represented by the following differential equation

$$\dot{x}(t) = Ax(t) \qquad \ldots\ 3.1(1)$$

Its solution gives a curve in the x-y plane. Such a curve is called the **trajectory** of the solution of 3.1(1) and it is represented by the graph which x(t) traces out.

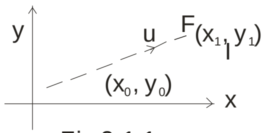


Fig 3.1.1

If, however, we wish to "force" (or "compel", or "control") the system to move from the point (x_0,y_0) to another point (x_1,y_1) we need to impose some function **u** which can **contro**l the movement of the particle described by the differential equation 3.1(1) so that it moves to the desired point (x_1,y_1) (see Fig.3.1.1). If this pre-desired movement is to be achieved, we affix u to the equation 3.1(1). To indicate the possible success of this process. The new equation will then be of the form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \end{aligned}\right\} \qquad \ldots 3.1(2)$$

This new equation 3.1(2) is then called a **control system** simply because it contains the control function u . The point (x_0, y_0) from which the dynamics takes off is called the **initial point** and the desired destination is called the **target (point)** of the system. The set of all the target points of the system is simply called the **target.**

Matrices The role of matrices in the analysis of control systems is of utmost importance. An equation of the form 3.1(1) is clearly a first order equation. Its solution can be fairly easily got even if it contains an additional term (u, for instance) as we will like it to contain. In that case it assumes the form 3.1(2). The machinery for getting the solution is called the method of **variation-of-parameters** which is very familiar to most people. The matrix method enables us to convert an equation of any order to a first order equation of the form 3.1(1) which "ipso facto" is solvable by the method of variation-of-parameters. For instance, the following second order differential equation.

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$$

can easily be converted to

$$\dot{x}(t) = Ax(t)$$

if we take $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and A = $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$.

Now if n, m, are integers we normally have the control system 3.1(2) to be of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and defined in E^n. In this equation, x is an n-vector, A some nxn matrix, B an nxm matrix and u an m-vector measurable function. We prefer to have the general system to be defined in so that we can deal with it in the Euclidean space of any dimension n. The control function u is assumed to take values in some m-dimensional Euclidean space E^m which is then said to be **admissible** and takes the system 3.1(4) from its initial point (x_0,y_0) to the target $(x_1,y_1,)$.

Important Assumption The control system 3.1(2) is first and foremost a differential equation whose solution is readily available, at least by the use of the variation-of-parameters. So here we are not interested in getting the solution precisely because it is always relatively well known and assumed to exist. Therefore, the crucial point at issue here is the determination of the **behaviour** of the system; nothing else is required.

History The subject of control theory is relatively new. It started about the middle of the twentieth century (around 1940 to 1952) and it is concerned primarily with the behavioural patterns of a system which is subjected to an appropriate admissible control imposed on it. Now, by the famous variation-of-parameters, the solution of the equation of control 3.1(2) is given by

$$x((t, x_0, u) = X(t)x_0 + X(t)\int_0^t X^{-1}(s)Bu(s)ds \qquad \ldots\ 3.1(3)$$

where X(t) is its **fundamental matrix** and represents its solution for u = 0. The fundamental matrix plays a very important role in the over-all analysis of differential systems. If the system under study is a delay equation, its fundamental matrix will be denoted by **X (t – s)** for suitable s. Although the solution 3.1(3) is expressed as x(t , u), for notational convenience we can denote it simply by x(t), that is, we can set

$$x((t, x_0, u) \text{ o } x(t) \quad \ldots \quad 3.1(4)$$

Now, in simple non technical terms, the control system 3.1(2) is said to be **Euclidean controllable** (that is, controllable in Euclidean space) if given some admissible control u the solution x(t) of the system is steered from the initial point x_0 to a final point x_1 in finite time t. Symbolically, we can write this thus: for each xo and each x_1 there exists an admissible control u and finite time t_1 such that the solution x(t), say, of 3.1(2) satisfies x(0)=xo and $x(t_1)=x_1$. If x_1 in this definition is the zero point $x_1 = 0$, we say that the given system is Euclidean **null-controllable.** Null-controllability is very important as we shall explain in the sequel. We will not go into the technicalities associated with controllability. However, R. E. Kalman [17] provided a criterion for the determination by calculation of the controllability of systems of the form 3.1(2). Kalman's theorem states that any control system of the type 3.1(2) is controllable if and only if the rank of the matrix $[B \ AB \ A^2B \ldots\ldots A^{n-}B]$ is equal to n. Hence, following Kalman, a second order control system in E^2 (after reduction to the form 3.1(2)) is controllable if and only if rank $[B \quad AB] = 2$; and a third order control system is controllable if and only if rank $[B \quad AB \quad A^2B] = 3$; and so on.

We are emboldened to use Kalman's theorem for a control system of any order because as we saw earlier, a differential equation (or a corresponding control system) of any order is reducible to 3.1(1) (or 3.1(2) as the case may be).

However, much as Kalman's theorem is convenient and easy to demonstrate with concrete examples, there is another qualitative criterion for determining controllability which Chukwu and Silliman [3] established which is very elegant in application in the analysis of controllability. It is used frequently and we used it in one of the works to be presented here even if in a skeletal form in this lecture. I hereby designate the Chukwu-Silliman-controllability criterion by C – S – C and it is stated as follows:

C – S –C … The control system $\overset{\times}{x} = Ax + B$ is Euclidean controllable in the interval [$0,t_1$] if and only if for t in [$0, t_1$] and $q \varepsilon E^n$ **,** $q^T x(t)B=0$, implies that q = 0, where T is matrix transpose.

Both the Kalman's theorem and C – S - C above show that it is not every system which has the function u attached to it that is in fact controllable.

Input-output relationship. Often, the control system can be described in terms of the so- called input-output relationship. A good example is the common battery charger system (Fig. 3.1.2). The charger is endowed with some mechanism (which could be some chemical acting as the control u).

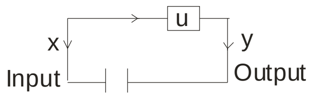


Fig 3.1.2

When a battery with low current is connected to the charger and the electricity supply E is switched on, the charger acts on the current in the battery called the **input x** and converts (or augments) it so that what emerges is called the **output y.** As a rule y is usually different from x. This difference is due to the action of the control function. In a more complicated system, a monitoring mechanism called the **feedback** is installed to report, as it were, on the efficiency of the system.

3.2. Applications Here we point out some of the common applications of the control function. The first is the battery charger of Fig. 3.1.2 just discussed. It helps to reactivate the life span of most rechargeable batteries.

Also, the cistern of the toilet is a contrivance which serves man. As a result of some mechanism installed in it, the cistern stops the inflow of water when the tank is full and allows more water when it is empty. This cycle is repeated again and again according to man's needs.

The brake system in cars – though it appears simple in construction and application – is made with some element of control in it which helps in safeguarding life inside and outside the car.

The electric kettles, those that switch themselves off automatically when the contents are sufficiently heated up, operate as they do on account of some control components installed by the manufacturers.

The traffic lights operate as they do as a result of the control principle employed during their manufacture and installation.

In a bad undesirable system, a control function can be designed which steers that very system to zero in finite time – a case of null-controllability. A good example is the power outage from NEPA. If we denote such outage by x, then a device of the form 3.1(2) and such that after some time x is constrained to hit zero (or vanish0, thus bringing the power outage to an end (that is in finite time) to man's delight; this is null-controllability in action..

Many other examples abound in which man harvests lots and lots of advantages as a result of the application of the principles of control theory. I will move to discuss briefly three of our earlier works in control theory. Slight mention will be made of the first two whereas the third (and last) one will receive a little bit more attention since the base equation is a neutral system. The problem solved in such examples often resembles the practices of man in the manufacture of modern war machines from the branch of control systems theory known as **game theory** which relates **pursuit** (for the attack) and **evasion** (for the defence), a relationship which is maximally utilized by both sides of any armed conflict.

3.3 Contributions to Control Theory. Here I present three selected works of mine. The first is on null-controllability [8] In it we employed the Leray-Schauder fixed point theorem to prove null-controllability for linear systems in Euclidean space. Although the system has some nonlinear perturbation, nevertheless null-controllability was established by means of the Leray-Schauder fixed point technique. The work was done about the time when the rinderpest disease was clearing cattle in the northern part of Nigeria. We proposed a system in which , if all the relevant conditions are satisfied, the problem solved would help to reduce to zero, perhaps, the number of cows that would perish after some time – this is null-controllability.

The second was on **total controllability** [12]. Total controllability of a system occurs if whenever the system hits a target, it remains within the target forever afterwards. The equation studied in the paper is completely nonlinear. Total controllability was established by perturbation methods using a new approach to nonlinear variation-of-parameters due to Lord and Mitchel [24].

The third one concerns cores of targets H ,say. The core of the target H, denoted by core(H), is the set of all the initial points $\{x_{oi}\}$ i=1,…,n of a system which generate corresponding solutions of the given system that belong to H. The qualitative conditions of convexity and compactness of the cores of a given target H of a neutral control system defined in Euclidean space are studied. This study appears to be very useful in both attack and defence in military strategy. Two crucial concepts are utilized in the proof of the main result of this work. The first is **asymptotic direction** – a concept in convex set theory. This very concept has a critical bearing on the problem at hand. For, let H be a target set of the system under study and suppose H contains M some mxn matrix. Suppose further that the fundamental matrix of the neutral system under study is X(t – s) for some s. It was first proved as a supporting lemma to the main theorem of the paper that if a point **a** belongs to core(H) then **X(t – s)a** belongs to H and conversely. Besides, it is known that a convex subset of the Euclidean space is bounded if and only if zero (0) is its only asymptotic direction. The second concept is based on the fact that M belonging to the target H is **weakly compact**. Under these conditions, it is then proved that the core of the target H, core(H), of the neutral control system given by

$$\dot{x}(t) - A\dot{x}(t-h) = Bx(t) + Cx(t-h) + Du(t)$$

… 3.3(1)

is compact if and only if the following system

$$\dot{x}(t) - A\dot{x}(t-h) = B^T x(t) + C^T x(t-h) + M^T u(t)$$

… 3.3(2)

is Euclidean controllable.

In the systems above, x is an n-column vector; u is an m-row vector; A, B, C each is an nxn constant matrix; D is an nxm matrix; M is an mxn matrix; and T denotes matrix transpose. D belongs to the system 3.3(1) under consideration and M belongs to the target. Therefore, the result can be viewed as being important in some sense to both sides of the conflict if utilized for military strategy.

We shall now wish to briefly discuss some known relationships, however tenuous, between some scientific studies (in particular Mathematics and Physics on the one hand) and another discipline.

4. EXTRA – SCIENTIFIC RELATIONSHIPS

In this concluding section, I wish to draw attention to a surprising relationship between Mathematics and a subject which does not belong to the realm of science at all. As a corollary, we shall discuss in somewhat greater detail a relationship somewhat tenuous, however, between Physics and that same subject.

It is interesting to note that it was reported [26] that the meeting of the French Philosophical Society of 1922 was well attended by both Physicists, Mathematicians and Philosophers alike. What they discussed in that meeting is not important to us. That reported 1922 joint meeting has now motivated me to discuss the noticed interest in Mathematics and Philosophy by some Mathematicians on the one hand, and the interest in Physics and Philosophy by some Physicists on the other.

4.1 Mathematics and Philosophy. There is a course even in the curriculum of our University's Philosophy Department which seems to (and in fact_does) have some strong resemblance to the concepts in Mathematics which are most frequently used in teaching some elements of **set theory** in Mathematics. The course in Philosophy is **Symbolic Logic** (PHIL 232).

However, one twentieth century mathematician appeared to be equally versed in his knowledge of mathematics as well as philosophy. He was an Englishman – **Bertrand Russell** (1872 - 1970). It was said of him even when he was alive that most of his contemporaries in Mathematics never knew that he was also a philosopher. In the same way, most of the philosophers of his time were unaware that he was also a consummate mathematician. Russell was a unique person indeed.

4.2. Physics and Philosophy.

Perhaps, overwhelmed by the demands of research, Albert Einstein once declared [26]:

"Ich muss in den Sternem suchenn
was nur auf Erden versagt ist".

which translates to

" I must search in the stars for what
is denied me on earth".

Einstein is very well known especially in the myth surrounding the development of the atomic bomb. He had a contemporary – Neils Bohr who cooperated with him in scientific research. Along with Max Planck also the two constituted an unforgettable trinity in the history of Physics. Working independently, the three – Max Planck (1858 - 1947), Albert Einstein (1879 - 1955), and Neils Bohr (1885 - 1962) made outstanding discoveries in various aspects of Quantum theory (of Physics) and so they jointly brought about the transition in the twentieth century physics from *Classical*_to *Quantum Theory* of Physics. Bohr was from Denmark; the first two are of Jewish origin.

Bohr and Einstein had many things in common and were personal friends. Both won the Nobel prize in physics in 1922 (Einstein's own was backdated to 1921). Although Neils Bohr was known to be an acknowledged physicist, he was also known to be a consummate twentieth century philosopher [26]; maybe it was their friendship that made Albert Einstein interested in philosophy. Thus, what Bertrand Russell was to Mathematics-Philosophy, Neils Bohr was to Physics-Philosophy

At this point I find it necessary to talk a little bit about that enigmatic figure and a contemporary of Neils Bohr who had a tenuous interest in philosophy too – Albert Einstein. We will highlight his misunderstood role in the development of the atomic bomb.

Albert Einstein Einstein was born in Germany in 1879 of Jewish parentage. A scientist of extra-ordinary capabilities, he taught in various Universities in Germany and Europe. Besides, he was also very outspoken on various issues - political, philosophical, moral. By the time he won the Nobel prize he held the citizenship of Germany and Switzerland simultaneously. In 1922 the German Foreign Minister – Walther Rathenau – also a Jewish German and Einstein's bosom friend was assassinated. There were very strong feelings that Einstein's assassination might also follow. Hence, Einstein shortly left Germany for Europe. By 1933 he went to settle permanently in the U S (Institute for Advanced Studies, Princeton). He was versatile and outspoken. Apart from Quantum

Theory, he also discovered both the **Special Theory of Relativity** and the **General Theory of Relativity.** On account of these discoveries and also his incisive comments on various current issues, the press gave him much publicity. Einstein also discovered the energy equation

$$E = mc^2$$

where **c** is the speed of light which is about 300,000 kilometres per second.

The Atomic Bomb In 1938 two scientists in Germany carried out a radiochemical analysis which involved the collision of neutrons and uranium chemical. It was found that after the collision some uranium remained which reacted again with further neutrons. And so on. Besides, after each collision an enormous amount of energy was produced. This kind of series is called a **chain reaction.** Theoretically, the chain reaction should continue in infinite sequences of reactions each of which generates lots and lots of energy. That experiment in Germany took place in 1938 just before the outbreak of the second world war. The two scientists who performed that historic experiment were **Otto Hahn** and **Fritz Strassman**.

Originally, Einstein was skeptical about the above result. But as Germany was then producing uranium in large quantities, Einstein recalled that experiment. So as the second world war was going on he suspected that Hitler might have started to produce the bomb based on the above principle of chain reactions. So he

wrote the American President F. D. Roosevelt. It was only after Einstein's reminder that Roosevelt in 1941set up a Board on the Bomb. In August 1945 America under President Truman (Roosevelt having died earlier in April) an atomic bomb was released on Hiroshima (Japan) and its devastating effects helped to end the second world war immediately. However, as a pacifist, Einstein had opposed war of any kind. He was known to have been criticizing the weakness of the League of Nations in securing peace in the world. Earlier (in 1931) Einstein was reported to have lamented the use of scientific equipment to the detriment of man, and thus
had declared [26]:

> **"In times of war applied science has given men the means to poison and mutilate one another. In times of peace, science has made our lives hurried and uncertain. Instead of liberating us from much of the monotonous work that has to be done, it has enslaved men to machines; men who work long wearisome hours mostly without joy in their labor and with the continual fear of losing their pitiful income".**

Soon after the horrible use of the atomic bomb various arguments were raging as to the origin of the bomb. Those arguments continued for very long.. Almost everybody insisted that Einstein was the direct manufacturer of the bomb. An issue of the Time Magazine in 1946 [29] had this to say:

> **"Einstein was the father of the bomb in two important ways: (1) it was his initiative which started US bomb research; (2) it was his equation ($E = mc^2$) which made the atomic bomb theoretically possible".**

Earlier, Einstein, the pacifist, made a personal denial about his authorship of the bomb. Barely three months after the war he published in the Atlantic Monthly [1] and stated as follows:

> **"I do not consider myself the father of the release of atomic energy. My part in it was quite indirect. I did not, in fact, foresee that it would be released in my time. I believed only it was theoretically possible".**

With the above statement from Einstein himself, the misconception about the origin of the atomic bomb is clearly closed beyond all reasonable doubt.

Let us conclude by focusing our attention on Bohr's and Einstein's position in Philosophy. Bohr was an established philosopher. However, Einstein, from his statements and relationship with Bohr, was merely interested marginally in philosophy. It would appear that his attitude adversely affected Bohr's later conception about philosophy. Clearly, to the extent that that philosophical studies helped to sharpen their intellect for use in their scientific pursuit, to that extent they appeared to like philosophy. Besides, the following contention of Bohr seems to suggest that their liking was for philosophy as a subject of study while at the same time they did not hold philosophers in very high esteem [26]. At times Bohr used the words *scientist* (literally, *physicist* and *expert* interchangeably. His thinking about philosophers is embodied in the following funny statement [26]:

> **"The relationship between scientists and philosophers is a very curious kind… . The difficulty is that it is hopeless to have any kind of understanding between scientists and philosophers directly".**

He concludes:

> **"What is the difference between an expert and a philosopher? An expert is someone who starts out knowing something about some things, goes on to know less and less , and ends up knowing everything about nothing, Whereas a philosopher is someone who starts out knowing something about some things, goes on to know more and more, and ends up knowing nothing about everything".**

CONCLUSION

I have tried to establish how research works in delay and control in differential equations have produced (and continue to produce) monumental results which in no small measure have contributed to man's over-all development. These narrow research areas can indeed be applied in other fields, thus forming strong, formidable inter-disciplinary co-operations with others who seek to improve man's life on earth.

Mr. Vice-Chancellor Sir, ladies and gentlemen, I am done. I appreciate your giving me your precious time.

THANK YOU ALL.

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